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AN INVESTIGATION OF THE NUMBER OF REGIONS CREATED BY THE CHORDS IN A CIRCLE

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ABSTRACT

This paper is an investigation of the number of regions created by the chords in a circle. Chords are drawn in circles with 1-5 points to find the number of regions for each circle. The relationship between the number of points and the number of regions in circles with 1-5 points is shown in a table, and a formula is created based off the powers of 2. The formula is rejected when the number of regions in a circle with six points does not equal the value from the formula. Using the calculator, a quartic regression is found, and is later tested to be successful with circles that have 1-8 points. Combinations, quadrilaterals, Platonic Solids and Euler's Formula are introduced, and used to determine the number of regions. A pattern between Pascal's Triangle and the number of regions is found. Two attempts are made to explain the relationship, but they are unsuccessful. In the addendum, a third, yet successful, attempt is made to prove the relationship algebraically.

PROBLEM STATEMENT

We will be exploring problems that involve the number of points on a circle and the regions in a circle. By altering the number of points that are drawn on the circle, we can also change the number of regions that are created from chords that connect the points of the circle.

Let’s take a look at this circle.

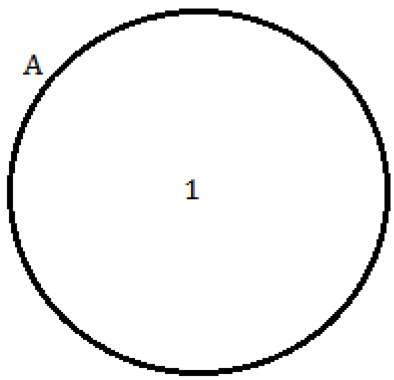

Fig. 1. Showing a point on a circle.

This circle has one point labeled, which means that this circle will only have one region inside of it. But what if there are two points drawn? This is shown in Figure 2.

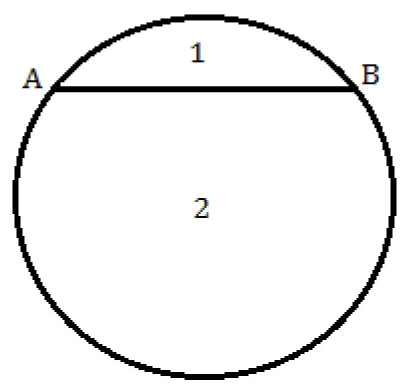

Fig. 2. A chord, $\overline{AB}$, has been formed in the circle.

As shown in the picture, there are two regions in a circle when there are two points drawn. What if we have three points? This can be seen in Figure 3.

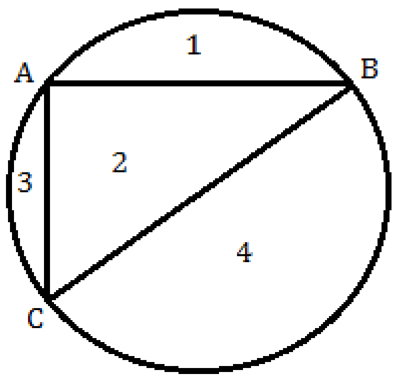

Fig. 3. Two other chords, $\overline{AC}$ and $\overline{BC}$, are labeled, and there are four regions in the circle.

From these pictures, there are some questions that can be formed:

- How does the number of points drawn in the circle affect the number of regions in the circle?
- Is there a pattern between the number of points and the number of regions?

RELATED RESEARCH

Before, we have seen circles with one through three points drawn. But now, we will look at circles with more than three points. Take a look at Figure 4.

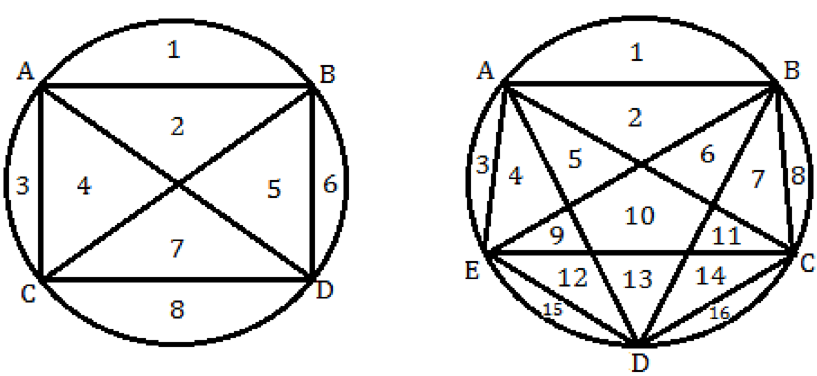

Fig. 4. These circles have four and five points labeled, and more chords are formed.

In both of these pictures, we can see that the chords are intersecting, but three chords should *not* intersect at one common point.

Is there a relationship between the number of points and the number of regions in a circle? Let's make a chart and see what happens to the number of regions as the number of points increase. The results are shown in Table 1.

Table 1. The Number of Regions in a Circle as the Number of Points Increase

Number of Points	Number of Regions
1	1
2	2
3	4
4	8
5	16
...	...
n	?

In the chart, we can discover that the number of regions in the circle increases exponentially. The number of regions are powers of two since

$$2^0 = 1, 2^1 = 2, 2^2 = 4, 2^3 = 8 \; and \; 2^4 = 16.$$

However, if we look closer, we can see that the exponents in the powers of two are one less than the number of points. Therefore, the formula to get the number of regions can be represented as

$$2^{n-1}$$

Now that we know this pattern, we can use this to predict the number of regions in circles with more than five points. Let's try a circle with six points!

Substitute six in for the number of points.

$$2^{6-1}$$

Simplify.

$$2^5 = 32$$

From this, we are going to predict that the number of regions in a circle with six points is 32. Let's see if our prediction is correct! Take a look at Figure 5.

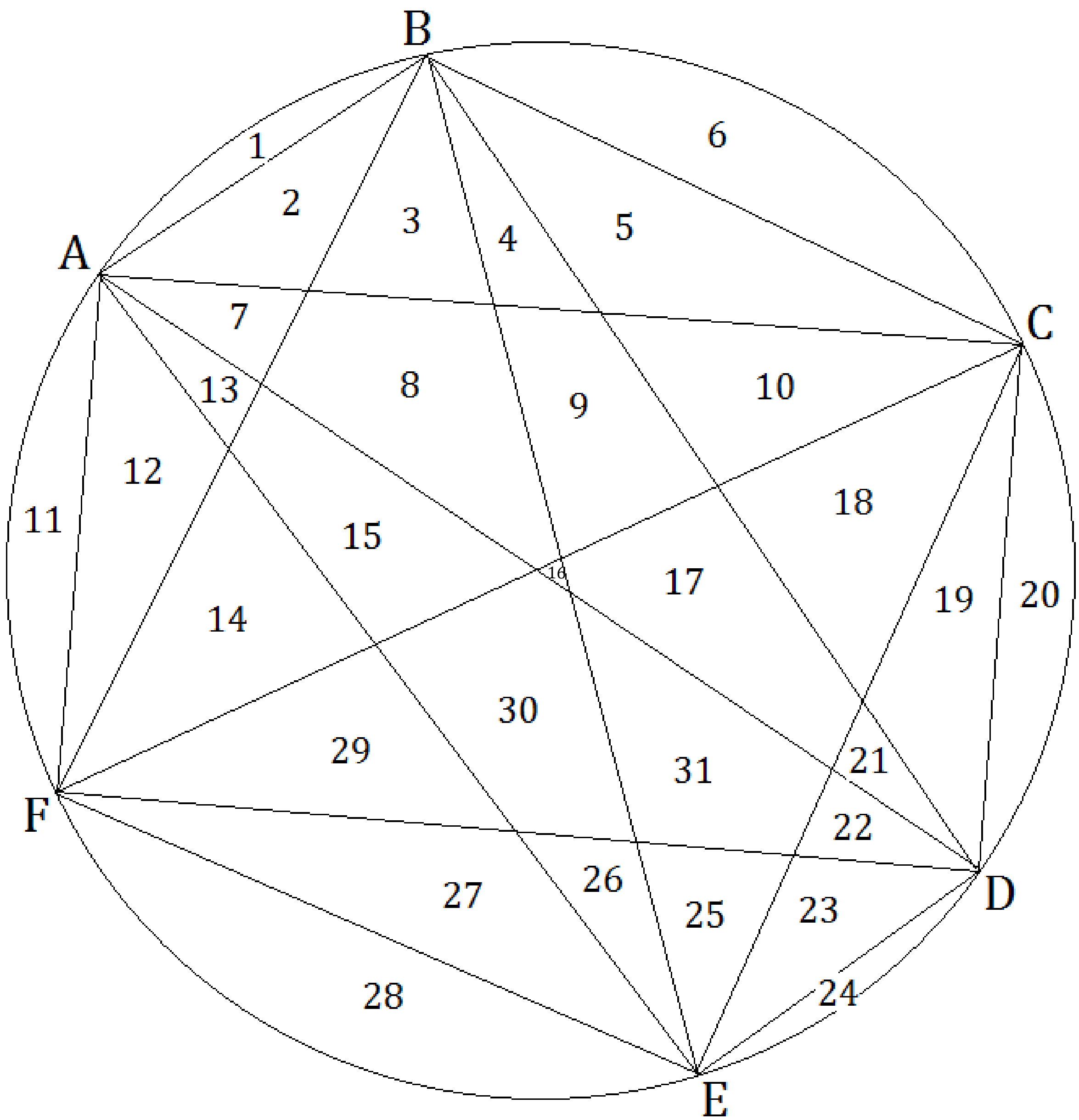

Fig. 5. The circle with six points has 31 regions.

As we can see, our prediction is not correct since a circle with six points has 31 regions! This shows how the pattern we found before is *not* true for circles with more than five points. We are now going to try and find another formula that will work for this case. We can plot the number of points and regions for circles with 1-6 points in the calculator. To get a table of values, we can press STAT from the home screen, and click on Edit, which is shown in Figure 6.

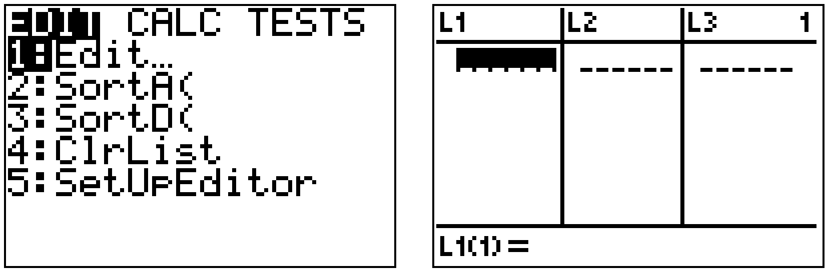

Fig. 6. Getting a table of values using the calculator.

We can fill in the table of values, and after that we can press STAT, move to CALC and scroll down to “QuartReg” since we are attempting to find the equation of the quartic regression curve. After pressing “QuartReg”, we can type in the columns that are used in the table, which are L_1 and L_2. This is shown in Figure 7.

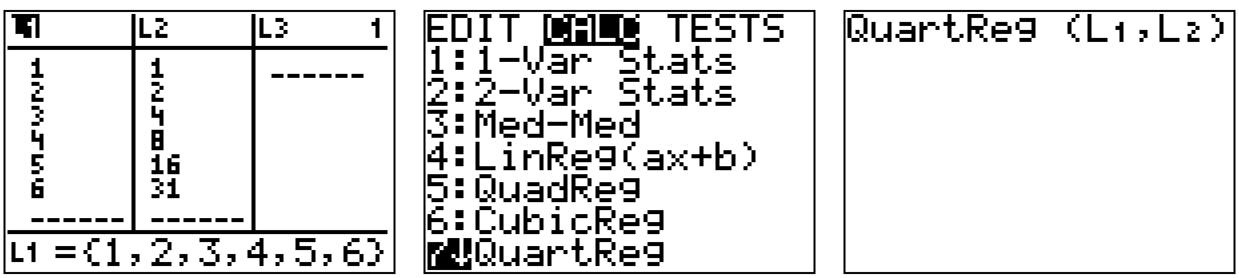

Fig. 7. Filling in the table of values and calculating the quartic regression.

Finally, we can press ENTER, and the values of each variable in the equation of the quartic regression curve are shown in Figure 8.

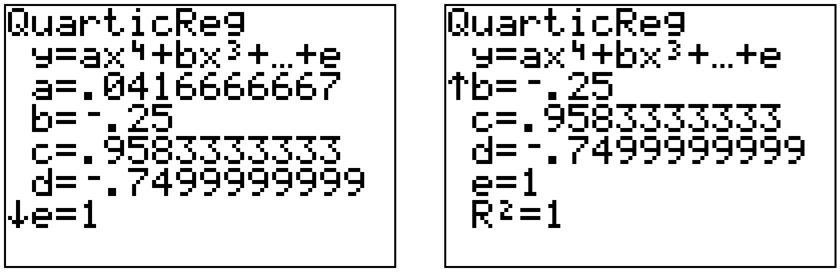

Fig. 8. For the quartic regression, R^2 is equal to one.

Since $R^2 = 1$, we can use this equation. After turning the decimals into fractions, we can see that the formula for the number of regions, R, is equal to

$$R = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$$

Let's see what happens when we have circles with seven and eight points. This is shown in Figure 9 and Figure 10.

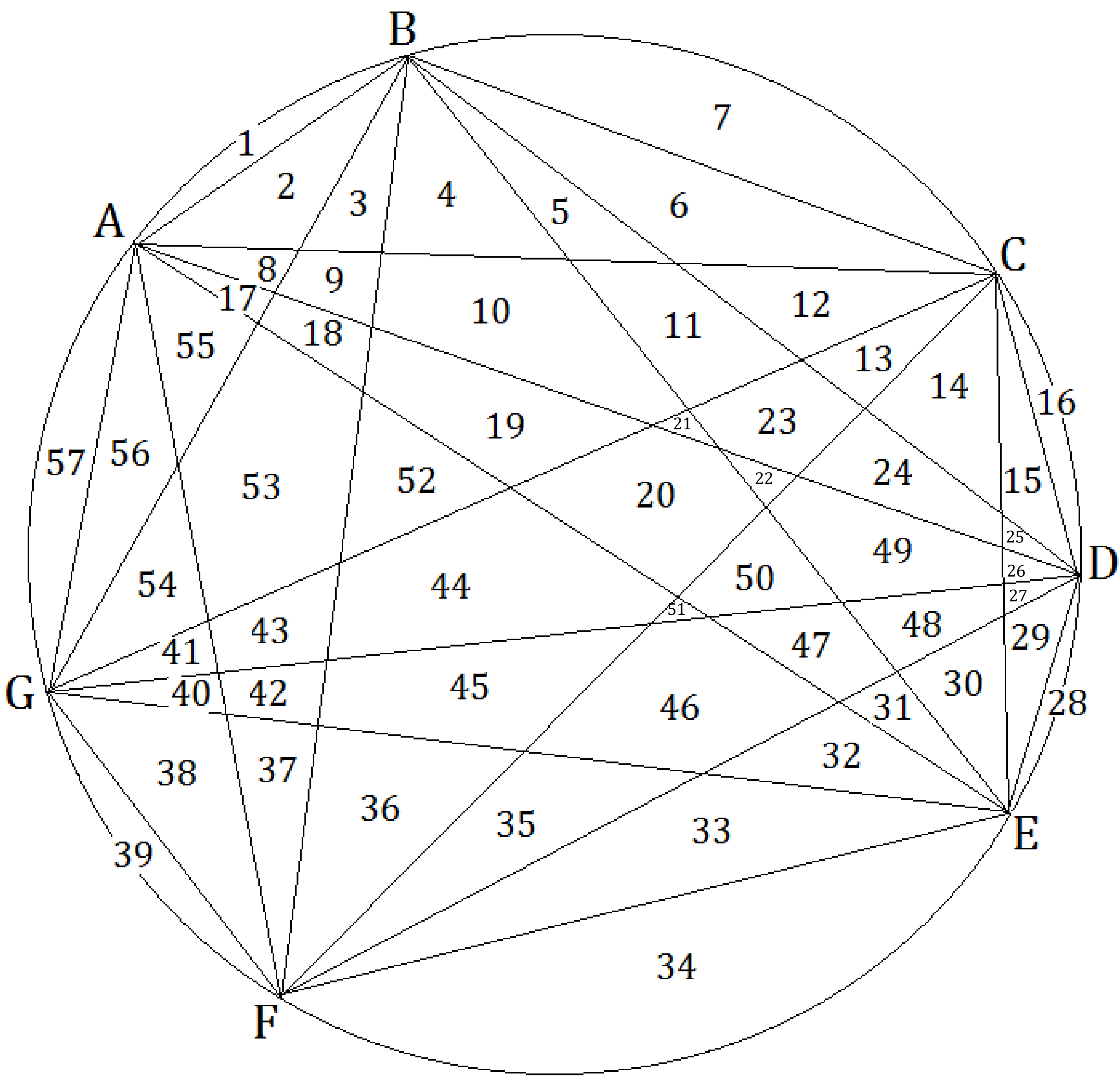

Fig. 9. The circle with seven points has 57 regions.

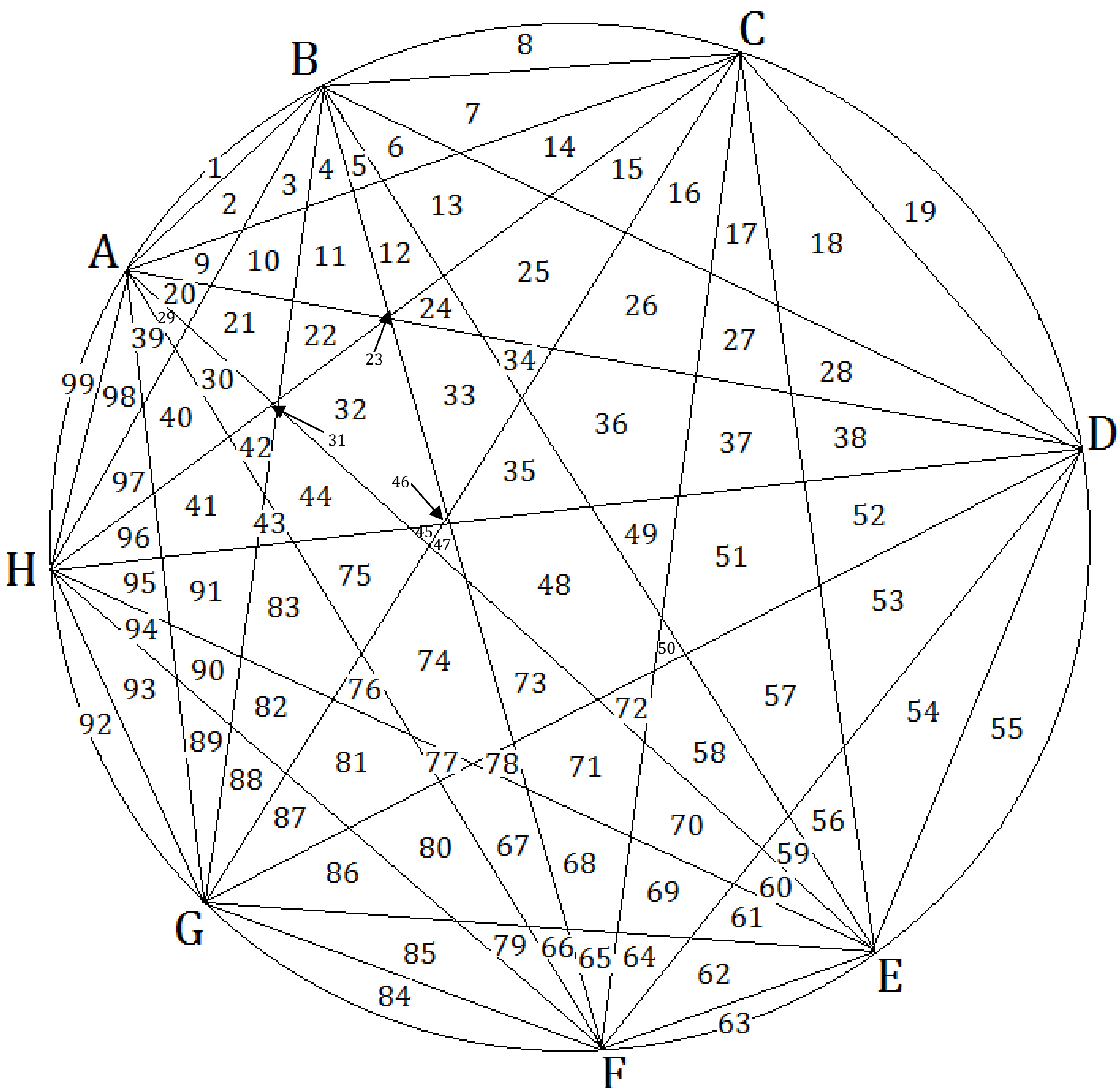

Fig. 10. The number of regions in a circle with eight points is 99.

In Figures 9 and 10, we can see that there are 57 regions in a circle with seven points, and 99 regions in a circle with eight points. Will these numbers be the same when using the formula? Let's make another table that shows the number of regions for these circles from the formula.

Table 2. The Number of Regions Using $R = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$

Number of Points on Circle	Number of Predicted Regions Using 2^{n-1}	Number of Regions Using R	Actual Number of Regions in Circle
1	1	1	1
2	2	2	2
3	4	4	4
4	8	8	8
5	16	16	16
6	32	31	31
7	64	57	57
8	128	99	99

From this table, we can see how these values are correct, which means that this formula works for circles with up to eight points.

We will now learn about combinations in the next section, which will help us continue to predict the number of regions in a circle as the number of points goes beyond eight points.

Combinations

In the previous section, we have found a formula that helps determine the number of maximum regions in a circle with 1-8 points. But what if we needed to find the maximum regions in a circle with ten or more points? We are now going to explore combinations, which are the arrangements of objects that are *not* in a specific order. Let's say we had a problem like this:

How many groups of four letters can be taken from A, B, C, D, E and F?

Now that we see this problem, we can list the number of groups of four that can be made without dealing with the order.

ABCD	ABDE	ACDE	ADEF	BCEF
ABCE	ABDF	ACDF	BCDE	BDEF
ABCF	ABEF	ACEF	BCDF	CDEF

As we can see, there are 15 possible arrangements that are created from this problem. Let's use our calculator and see if this is correct.

We can type "6" into the calculator since there are six people in this problem, and then we can press MATH, and move to PRB, or probability. This is shown in Figure 11.

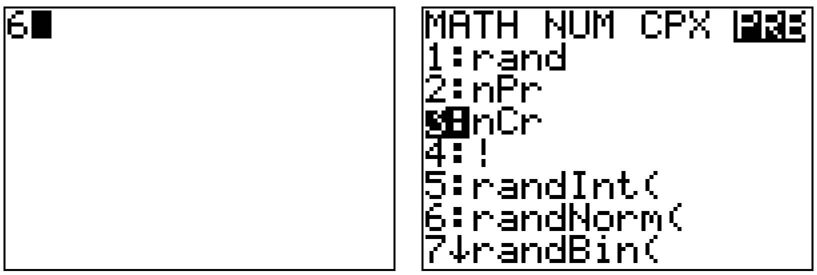

Fig. 11. Calculating the number of possible arrangements in a combination with 6C4.

After that, we can click on "nCr", which tells the calculator that this is a combination, and then press "4" since we had to create groups of four. After pressing ENTER, we get our result in Figure 12.

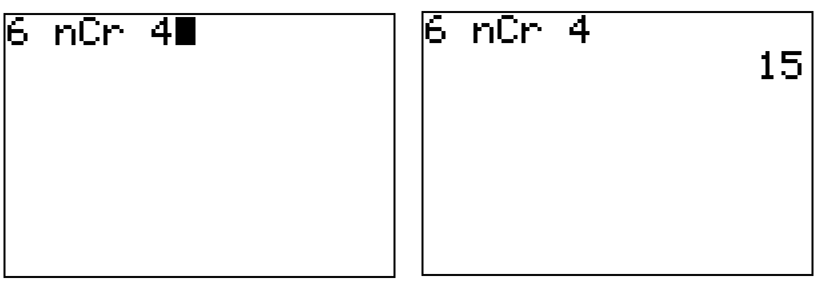

Fig. 12. Showing the number of combinations on the calculator.

We can see that this is correct for this problem!

Now, instead of listing the number of combinations and counting them or using a calculator, there is a formula that we can use to obtain this. The formula is

$$nCr = \frac{n!}{(n-r)!\,r!}$$

In this formula, the variable n represents the number of objects that are being used, and the variable r represents the number of objects in each group. We can take the problem from above and use this formula to get the number of possible arrangements.

Substitute 6 in for n and 4 in for r.

$$6C4 = \frac{6!}{(6-4)!\,4!}$$

Simplify.

$$6C4 = \frac{6 * 5 * 4 * 3 * 2 * 1}{2 * 1 * 4 * 3 * 2 * 1}$$

Since 6 is divisible by 3, the 2 in the denominator can cancel out, and the 6 on the numerator can be replaced with 3.

$$6C4 = \frac{3 * 5 * 4 * 3 * 2 * 1}{1 * 4 * 3 * 2 * 1}$$

Cancel out the 4, 3, 2 and 1 on the numerator and denominator.

$$6C4 = \frac{3 * 5 \cancel{* 4 * 3 * 2 * 1}}{1 \cancel{* 4 * 3 * 2 * 1}} = \frac{3 * 5}{1}$$

Simplify.

$$6C4 = 15$$

As a result, there are 15 different ways to make groups of four from six letters.

Since we understand what the number of combinations is in 6C4, we will take a look at how this combination will be applied to counting the number of vertices. Let's define some terms that will be used throughout this section. A **region** is an area inside a two-dimensional figure, as we have been determining the number of maximum regions in a circle. A **vertex** is a point that is formed from the intersection of two segments, and an **edge** is a side of a two or three-dimensional polygon.

The combination, 6C4, will help us count the number of interior vertices in a circle specifically with six points. Four of the six points will need to be used, making a quadrilateral. In the quadrilateral, the diagonals must intersect in order for an interior vertex to be created. In total, there would be 15 quadrilaterals inside of the hexagon, which we already know since we are using the combination 6C4. Three of these quadrilaterals are shown in Figure 13.

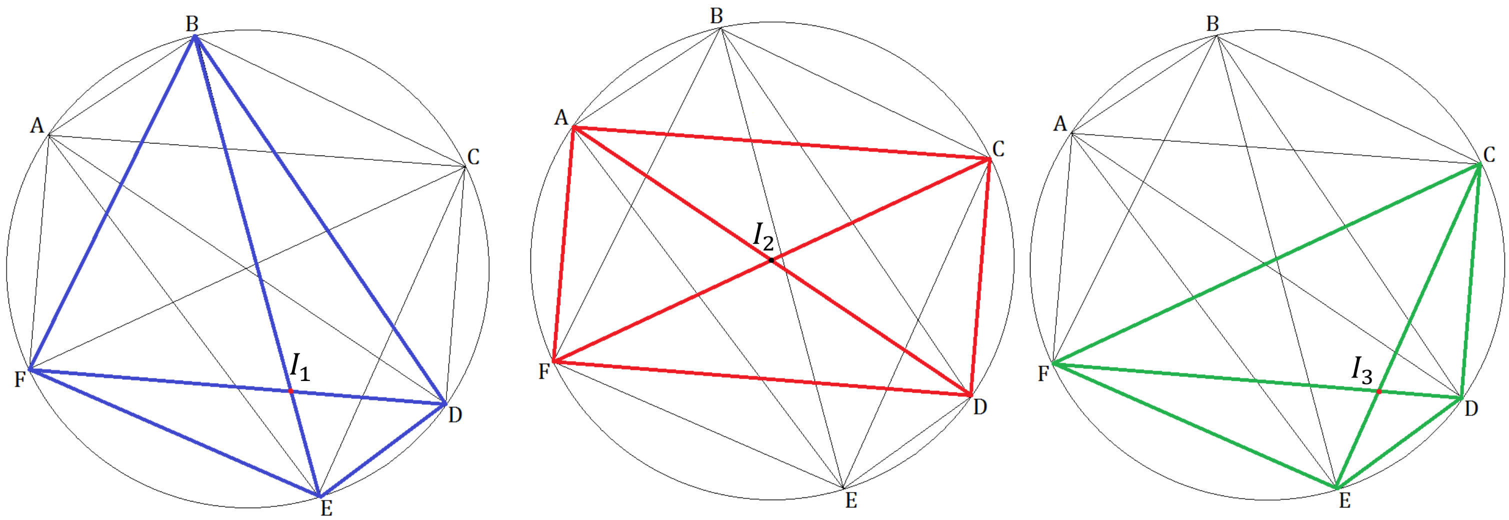

Fig. 13. The interior vertices of these quadrilaterals are represented as I_1, I_2 and I_3.

Each quadrilateral in the hexagon creates an interior vertex from the diagonals that intersect, and because there are 15 different quadrilaterals, there will also be 15 interior vertices.

Now that we understand how the combination 6C4 is used in a circle with six points, we will need to know about Platonic Solids, which consist of the tetrahedron, cube, octahedron, dodecahedron and icosahedron.

Finding the Total Number of Regions

Platonic Solids and Euler's Formula

Platonic Solids are polyhedra, or three-dimensional figures, that have straight edges, vertices and **faces**, which are congruent, regular polygons. The types of Platonic Solids are shown in Figure 14.

Fig. 14. There are only five known Platonic Solids that exist.

We are going to look at the number of edges, vertices and faces of each Platonic Solid, and then determine if there is a relationship between all of them. In Table 3 below, faces, edges and vertices are abbreviated as F, E and V.

Table 3. The Faces, Edges and Vertices on the Platonic Solids

Name of Solid	F	E	V
Tetrahedron	4	6	4
Cube	6	12	8
Octahedron	8	12	6
Dodecahedron	12	30	20
Icosahedron	20	30	12

From looking at Table 3, we can see that if we take the number of faces, subtract the number of edges from the number of faces and add the number of vertices to that difference, the answer will be 2 for all of the Platonic Solids. This is known as **Euler's formula**, and it can be displayed as

$$F - E + V = 2.$$

Euler's formula works with three-dimensional figures. However, we need to figure out how to make this formula work for two-dimensional figures since we are dealing with the number of regions in a circle. We are going to use a cube in this attempt and ultimately try to come up with an equation for two-dimensional figures. Let's start by forming the cube from a piece of paper, which is shown in Figure 15.

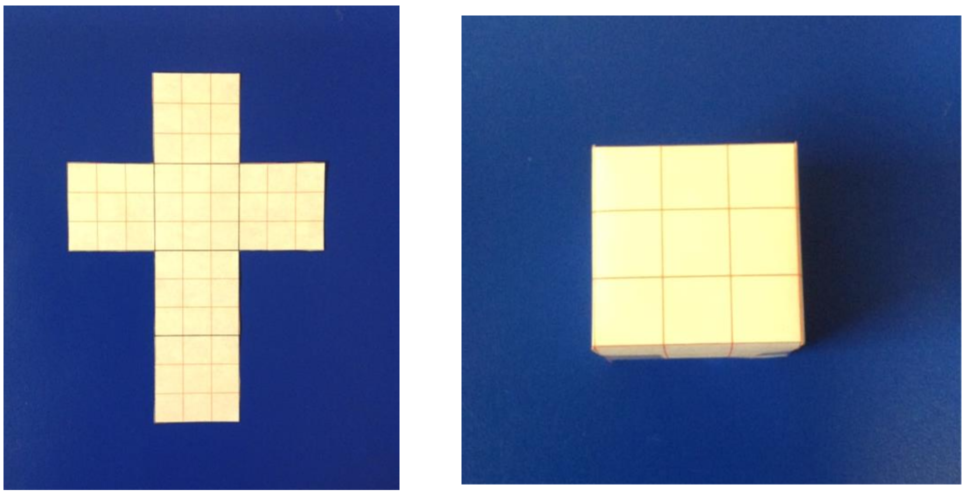

Fig. 15. Creating a 3x3 cube by folding the net of the cube and taping its edges.

We can "smash" this paper cube in order to make it a two-dimensional figure, and the flattened cube is shown in Figure 16.

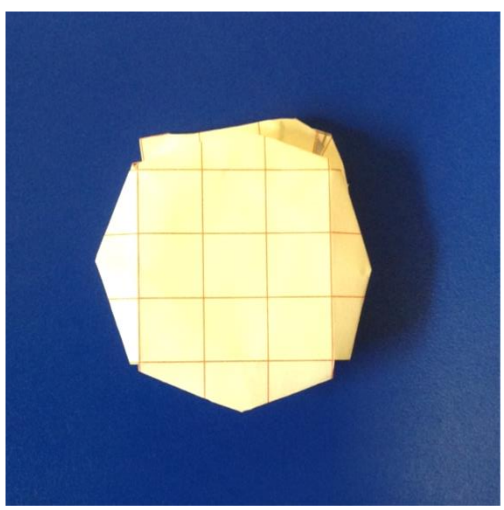

Fig. 16. Flattening the paper cube, but with the edges intact.

One of the faces, the bottom face, can be removed from inside of the cube because we have made this cube into a two-dimensional figure. But now, there are *five* faces rather than six, which means Euler's formula will no longer hold. We can see this in Figure 17.

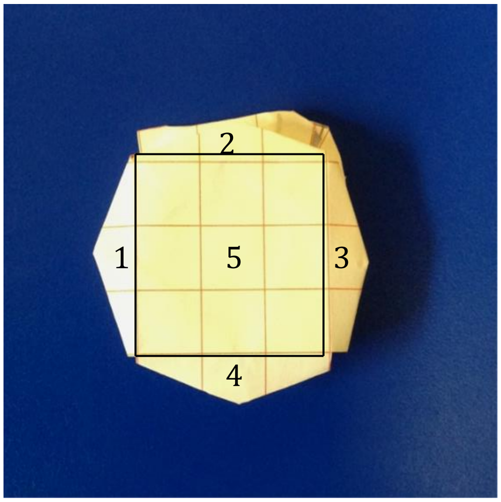

Fig. 17. The number of faces in this figure is five, unlike a cube that has six faces.

Now what happens to the removed face? This is shown in Figure 18.

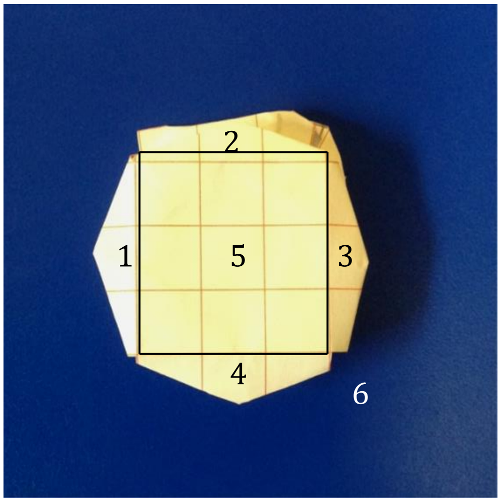

Fig. 18. The bottom face that has been removed becomes an exterior region.

The bottom face has been flattened enough to become a plane with an infinite length and width and no height. It is also no longer inside of the figure, which means the removed face turns into an **unbounded exterior region**, or a region that is not surrounded by any boundaries or edges.

However, we are only dealing with the *interior* regions of a two-dimensional figure, as shown in Figure 19.

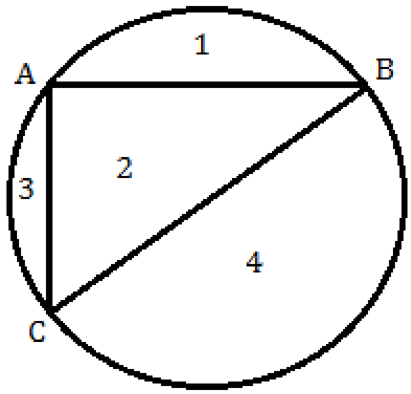

Fig. 19. Only the interior regions of a circle with three points are labeled.

Since we are exploring the interior regions of a circle and not the exterior regions, the unbounded exterior region can be ignored. This decreases our total in the new formula to 1 since the exterior region is not being included, making us still have five faces in the figure instead of six. As a result, the new equation can be shown as

$$R - E + V = 1.$$

We still have the variables E and V that represent the number of edges and vertices, but now we have the variable R instead of F, which shows the number of interior regions in a two-dimensional figure.

Now why did we not use the net of the paper cube? Let's take a look at Figure 20.

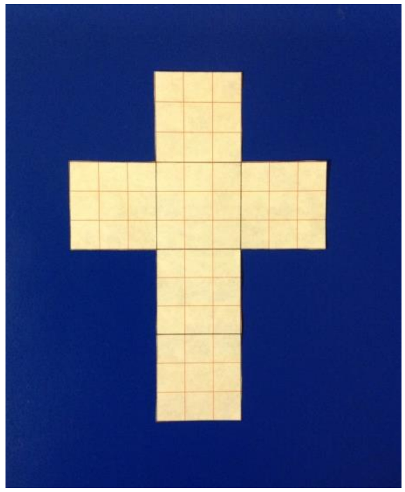

Fig. 20. Showing the net of the 3x3 cube.

Looking back at Table 3, there are 12 edges and 8 vertices on a cube. On the net of the cube, the number of faces is still six and the number of vertices is also eight, but the number of edges changes. Some edges that are connected together in the cube are separated in the net; while others do not change at all. These different edges are labeled in Figure 21.

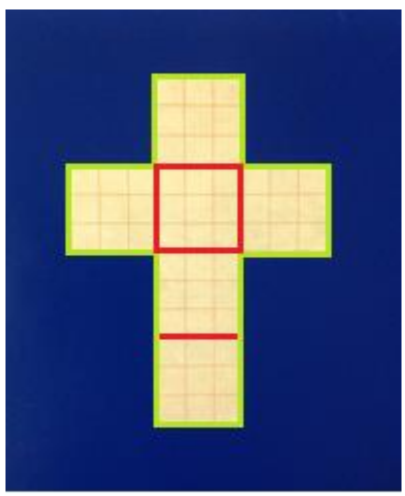

Fig. 21. The red edges remain the same; while the green edges connect when forming a cube.

The net in Figure 21 has 19 edges, which shows how there are seven "extra" edges on a net that connect when forming a cube. Using the net would provide us with the wrong equation. If we were to take the number of faces on the net, subtract it by the number of edges on the net and add the number of vertices on the net, the total would not equal 1 because the "extra" edges would end up being double-counted. Therefore, the net of the cube could not be used because the extra edges would affect the total that would be obtained. By creating the cube and smashing it, we are able to keep the same number of edges in the cube since it is not being altered.

We will now go back to finding the total number of vertices in a circle with six points. Before, we found the number of interior vertices, which was represented as the combination 6C4. An interior vertex is shown in Figure 22.

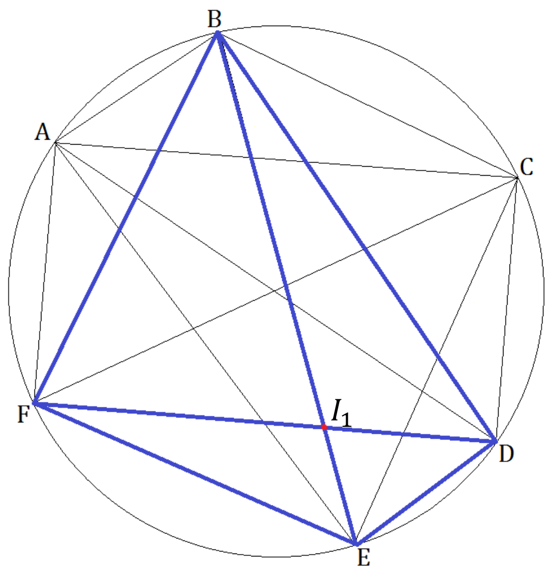

Fig. 22. One of the 15 interior vertices is made from the diagonals of a quadrilateral.

However, we also need the exterior vertices of the hexagon. The number of exterior vertices is equal to the number of points on the circle, meaning there would be six exterior vertices in this case. Since we have the number of exterior and interior vertices, we can add them together to get the total number of vertices, which is equal to 21. All of this can be substituted into the new formula we created.

$$R + (6C4 + 6) - E = 1$$

Simplify 6C4.

$$R + (15 + 6) - E = 1$$

Add 15 and 6 in the parentheses.

$$R + 21 - E = 1$$

If we had an unknown number of points on a circle that is represented as n, the total number of vertices would be equal to $nC4 + n$, which can be shown as

$$V = nC4 + n$$

The variable n is equal to the number of exterior vertices since it is the number of points on the circle, and the combination $nC4$ shows the number of interior vertices. The interior vertices would still be formed from the intersection of the diagonals in quadrilaterals, and the quadrilaterals use four of the n number of points. Substituting this value into the formula would change the equation to

$$R + (n + nC4) - E = 1$$

Now that we understand the number of vertices for a circle with six points and a circle with n points, we can move onto finding the number of edges for those circles.

Edges

For the circle with six points, there are six exterior edges since a hexagon is formed from the six exterior vertices. In addition to this, each of the vertices is an endpoint for five edges. This can be shown in Figure 23.

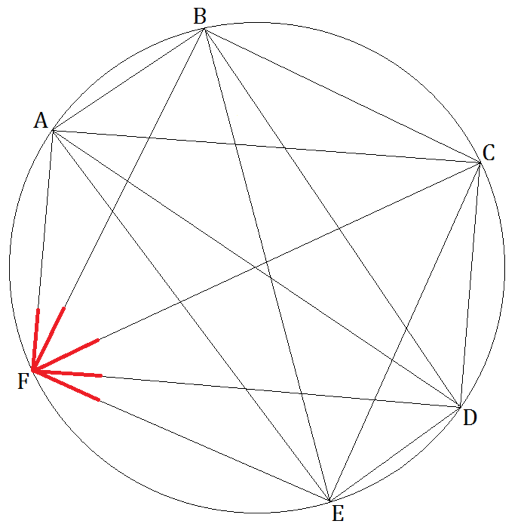

Fig. 23. Exterior vertex F connects five different edges.

From the information given, we can represent this as the product of six and five, and this is the first part of the total number of edges. The second part deals with the number of interior edges, which uses the interior vertices. As we have discovered before, there are a total of 15 interior vertices, which is shown as the combination 6C4. Each interior vertex is created by the intersection of four edges that are from the diagonals of the quadrilateral. The four edges that an interior vertex connects to are shown in Figure 24.

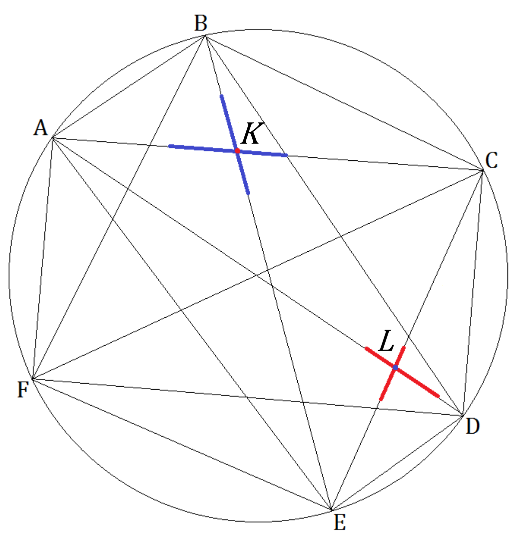

Fig. 24. Interior vertices K and L are endpoints of four edges.

Therefore, this part of our equation can be represented as the product of 6C4 and 4. Since we have the number of interior edges and the number of exterior edges, we can show this as

$$(6)(5) + (6C4)(4)$$

Although we have the "total number of edges", this expression still needs a third part added to it. In this case, all of the edges of the circle are double-counted. If we took the exterior edge $\overline{FA}$, for example, it could also act as the exterior edge $\overline{AF}$. There are two different names, but they still use the same edge. Since every edge is double-counted, we would have to divide the sum of the interior and exterior edges by two. As a result, the total number of edges in a circle with six points can be displayed as

$$E = \frac{(6)(5) + (6C4)(4)}{2}$$

We can also find the number of edges for a circle with an n number of points. There are n points on the circle, and the number of edges that one of those points would connect to would be represented as $n - 1$. As we saw in the circle with six points, an exterior vertex was the endpoint of five edges, and five is one less than six. Thus, the number of exterior edges would be equal to the product of n and $n - 1$, or

$$n(n - 1)$$

We have previously found the number of interior vertices when we were calculating the total number of vertices, which was equal to $nC4$. In Figure 24, we showed how an interior vertex is the endpoint for four edges since those four edges of one of the quadrilaterals make up the interior vertex. From this, the number of interior edges would be equal to the product of $nC4$ and 4.

The sum of the interior and exterior edges would have to be divided by 2 because all of the edges are double-counted, as we explained in the problem for a circle with six points. The total number of edges for a circle with n points would then have to be equal to

$$E = \frac{n(n-1) + (nC4)(4)}{2}$$

Since we have the number of vertices and edges for the circle with six points and the circle with n points, we will now plug these values into the formula to solve for the total number of regions for each circle.

Using the R Formula

We are going to determine the total number of regions for the circle with six points. The formula we left off with on Page 19 only had the total number of vertices, and was represented as

$$R + 21 - E = 1$$

But now, we can replace E with $\frac{(6)(5)+(6C4)(4)}{2}$ and solve for the interior regions using this formula.

Substitute $\frac{(6)(5)+(6C4)(4)}{2}$ in for E.

$$R + 21 - \frac{(6)(5) + (6C4)(4)}{2} = 1$$

Simplify the numerator of the fraction.

$$R + 21 - \frac{90}{2} = 1$$

Divide 90 by 2.

$$R + 21 - 45 = 1$$

Simplify.

$$R - 24 = 1$$

Add 24 to both sides of the equation

$$R = 25$$

We can see that the number of interior regions in a circle with six points is equal to 25. In order to get the *total* number of regions, however, we have to add the number of exterior regions, or the regions that are located outside of the hexagon. Since there are six exterior regions, we can add this to 25 to get 31 as the total number of regions.

We are now going to find the total number of regions for the circle with n points using the same method. The formula we had on Page 19 was shown as

$$R + (n + nC4) - E = 1$$

We can substitute in the number of edges, $\frac{n(n-1)+(nC4)(4)}{2}$, in this equation to get

$$R + n + nC4 - \frac{n(n-1) + (nC4)(4)}{2} = 1$$

Simplify $\frac{(nC4)(4)}{2}$ since 4 is divisible by 2.

$$R + n + nC4 - \left(\frac{n(n-1)}{2} + 2(nC4)\right) = 1$$

Distribute the negative to the terms in the parentheses.

$$R + n + nC4 - \frac{n(n-1)}{2} - 2(nC4) = 1$$

Combine $nC4$ and $-2(nC4)$ since they are like terms.

$$R - \frac{n(n-1)}{2} - nC4 + n = 1$$

Subtract n from both sides.

$$R - \frac{n(n-1)}{2} - nC4 = 1 - n$$

Add $\frac{n(n-1)}{2}$ and $nC4$ to both sides to isolate R.

$$R = \frac{n(n-1)}{2} + nC4 - n + 1$$

From this, the number of interior regions in a circle with n points is equal to

$$R = \frac{n(n-1)}{2} + nC4 - n + 1.$$

We can add the number of **segments** to this expression to get the total number of regions. A segment is the area that lies between a chord and a portion of the circle. In this case, there would be n segments because there are n points on the circle.

Add n to $\frac{n(n-1)}{2} + nC4 - n + 1$.

$$R = \frac{n(n-1)}{2} + nC4 - n + 1 + n$$

Eliminate n and $-n$.

$$R = \frac{n(n-1)}{2} + nC4 + 1 \qquad \text{(Equation 1)}$$

Equation 1 above shows the total number of regions, but we can simplify this even further to find another way to represent the total number of regions.

Using the formula, $nCr = \frac{n!}{(n-r)!r!}$, rewrite $nC4$ as $\frac{n!}{(n-4)!4!}$.

$$R = \frac{n(n-1)}{2} + \frac{n!}{(n-4)!\,4!} + 1$$

Now, when we have a factorial, we multiply the original number by numbers that are one less than the previous number until we reach one. For instance, 4! is equal to $4 * 3 * 2 * 1$, which is simplified to 24. However, with a variable, it can go on infinitely, so for $n!$, it would be

$$n(n-1)(n-2)(n-3)(n-4)\ \ldots$$

For this case, limit $n!$ to $n(n-1)(n-2)(n-3)(n-4)!$ and substitute it into the expression.

$$R = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!\,4!} + 1$$

Cancel out $(n-4)!$ on the numerator and denominator.

$$R = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{4!} + 1$$

Simplify 4!.

$$R = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{4 * 3 * 2 * 1} + 1$$

Simplify the denominator.

$$R = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{24} + 1$$

Distribute n to $(n-1)$ on both of the numerators.

$$R = \frac{n^2 - n}{2} + \frac{(n^2 - n)(n-2)(n-3)}{24} + 1$$

Multiply $(n^2 - n)$ by $(n-2)$.

$$R = \frac{n^2 - n}{2} + \frac{(n^3 - 2n^2 - n^2 + 2n)(n-3)}{24} + 1$$

Simplify.

$$R = \frac{n^2 - n}{2} + \frac{(n^3 - 3n^2 + 2n)(n-3)}{24} + 1$$

Multiply $(n^3 - 3n^2 + 2n)$ by $(n-3)$.

$$R = \frac{n^2 - n}{2} + \frac{n^4 - 3n^3 - 3n^3 + 9n^2 + 2n^2 - 6n}{24} + 1$$

Simplify.

$$R = \frac{n^2 - n}{2} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} + 1$$

Multiply $\frac{n^2-1}{2}$ by $\frac{12}{12}$ to make a common denominator.

$$R = \frac{12n^2 - 12n}{24} + \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} + 1$$

Simplify.

$$R = \frac{n^4 - 6n^3 + 23n^2 - 18n}{24} + 1$$

Factor out $\frac{1}{24}$.

$$R = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

Distribute $\frac{1}{24}$ to all of the terms in the parentheses.

$$R = \frac{1}{24}n^4 - \frac{6}{24}n^3 + \frac{23}{24}n^2 - \frac{18}{24}n + \frac{24}{24}$$

Simplify.

$$R = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$$

Therefore, the total number of regions for a circle with n points is also equal to

$$R = \tfrac{1}{24}n^4 - \tfrac{1}{4}n^3 + \tfrac{23}{24}n^2 - \tfrac{3}{4}n + 1.$$

This is the same expression we got when we obtained the quartic regression on Page 6. Since they are equal to each other, we can find the total number of regions for *any* circle using this expression. Let's go back to Equation 1 on Page 24.

$$R = \frac{n(n-1)}{2} + nC4 + 1$$

We can express Equation 1 using combinations. The combination nC4 is already in Equation 1, and 1 can be shown as the combination nC0 because $0! = 1$. Using the formula, $nCr = \frac{n!}{(n-r)!r!}$, we can see that

$$nC0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!\,0!} = \frac{1}{0!} = 1.$$

But how do we express $\frac{n(n-1)}{2}$ as a combination nCr? We are going to see what nC2 is equal to using the formula, $nCr = \frac{n!}{(n-r)!r!}$.

Substitute 2 for r into the formula, $nCr = \frac{n!}{(n-r)!r!}$.

$$nC2 = \frac{n!}{(n-2)!\,2!}$$

Rewrite $n!$ as $n(n-1)(n-2)!$.

$$nC2 = \frac{n(n-1)(n-2)!}{(n-2)!\,2!}$$

Eliminate $(n-2)!$ on the numerator and denominator.

$$nC2 = \frac{n(n-1)}{2!}$$

Simplify 2!.

$$nC2 = \frac{n(n-1)}{2}$$

Since $nC2 = \frac{n(n-1)}{2}$, it can replace the $\frac{n(n-1)}{2}$ in Equation 1. As a result, the new equation is

$$R = nC4 + nC2 + nC0 \qquad \text{(Equation 2)}$$

Now that we have Equation 2, we can see if it works when finding the number of regions in circles with a certain number of points. This is shown in Table 4.

Table 4. Using the $nC4 + nC2 + nC0$ Formula

n	R	$\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$	$nC4 + nC2 + nC0$
4	8	8	8
5	16	16	16
6	31	31	31
7	57	57	57
8	99	99	99

In Table 4, n represents the number of points on the circle, R shows the number of regions on the circle and the values from using the quartic regression and Equation 2 are shown in the last two columns. As we can see, the values from using Equation 2 are equal to the number of regions for each of those circles and the values from the quartic regression, so Equation 2 can be used as an alternative to the quartic regression. The second equation is easier to use than the regression, and it is also easier to plug into the calculator.

In the next section, we will look at a pattern in **Pascal's Triangle** that is applied to the number of regions in a circle with a certain number of points.

Using Pascal's Triangle

Before, we discovered a new formula to calculate the number of regions in a circle rather than using the quartic regression. We are now going observe a pattern in Pascal's Triangle relating to the number of regions. The following pattern is shown in Table 5.

Table 5. The Number of Regions and the Sum of the First Five Terms of Pascal's Triangle

n	R	Powers of 2	Rows of Pascal's Triangle	Sum of Rows of Pascal's Triangle	Sum of First Five Terms of Each Row of Pascal's Triangle
1	1	1	**1**	1	**1**
2	2	2	**1 1**	2	**2**
3	4	4	**1 2 1**	4	**4**
4	8	8	**1 3 3 1**	8	**8**
5	16	16	**1 4 6 4 1**	16	**16**
6	31	32	**1 5 10 10 5** 1	32	**31**
7	57	64	**1 6 15 20 15** 6 1	64	**57**
8	99	128	**1 7 21 35 35** 21 7 1	128	**99**

From Table 5, we can see how the sum of the rows of Pascal’s Triangle are equal to the powers of two, and the sum of the first five terms of each row of Pascal’s Triangle are the same as the number of regions, R, for each number of points on the circle. All of the terms are also used until the sixth row and beyond, where only five of the terms are then used.

Will this pattern continue beyond eight points? Look at Table 6.

Table 6. The Sum of the First Five Terms of Pascal’s Triangle in a Circle with 9-12 Points

n	R	Powers of 2	Rows of Pascal’s Triangle	Sum of First Five Terms of Each Row of Pascal’s Triangle	Difference Between Powers of 2 and R
9	163	256	**1 8 28 56 70** 56 28 8 1	**163**	93
10	256	512	**1 9 36 84 126** 126 84 36 9 1	**256**	256
11	386	1,024	**1 10 45 120 210** 252 210 120 45 10 1	**386**	638
12	562	2,048	**1 11 55 165 330** 462 462 330 165 55 11 1	**562**	1,486

As we can see, the pattern proves to exist for circles with 9-12 points. In addition to this, Table 6 shows the difference between the powers of two and the number of regions. What if we take the difference between the powers of two and R, and calculate the quotients of two consecutive numbers in the difference? This is shown in Table 7.

Table 7. Quotients for Every Two Consecutive Numbers in the Difference

n	R	Powers of 2	Difference Between Powers of 2 and R	Quotients
6	31	32	1	
7	57	64	7	7.00
8	99	128	29	4.14
9	163	256	93	3.21
10	256	512	256	2.75
11	386	1,024	638	2.49
12	562	2,048	1,486	2.33

Since the quotients of two consecutive numbers are relatively consistent, we can predict that this will have an exponential regression. Let's see if our prediction is correct using the calculator. The result is shown in Figure 26.

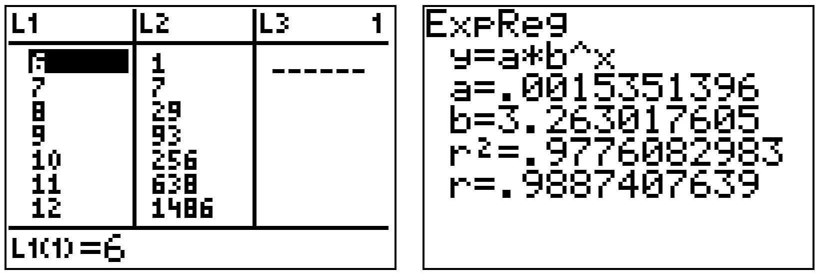

Fig. 26. For the exponential regression, the value of r is very close to 1.

Therefore, when calculating the regression between the number of points on the circle and the difference between the powers of two and R, we can see that is will most-likely have an exponential regression. However, since r is not equal to one, this regression cannot be used.

Pascal's Triangle and Combinations

We are still searching to explain the relationship between the combination formula for R and the first five numbers in Pascal's Triangle. We are going to attempt to use combinations since the formula we discovered for R on Page 28 was expressed using combinations. Recall the combination formula for R is

$$R = nC4 + nC2 + nC0.$$

We can write the rows of Pascal's Triangle as combinations. This is shown in Figure 27.

			n	R
Row 1 →	$\binom{0}{0}$	1	1	1
Row 2 →	$\binom{1}{0}\binom{1}{1}$	1 1	2	2
Row 3 →	$\binom{2}{0}\binom{2}{1}\binom{2}{2}$	1 2 1	3	4
Row 4 →	$\binom{3}{0}\binom{3}{1}\binom{3}{2}\binom{3}{3}$	1 3 3 1	4	8
Row 5 →	$\binom{4}{0}\binom{4}{1}\binom{4}{2}\binom{4}{3}\binom{4}{4}$	1 4 6 4 1	5	16
Row 6 →	$\binom{5}{0}\binom{5}{1}\binom{5}{2}\binom{5}{3}\binom{5}{4}\binom{5}{5}$	1 5 10 10 5 1	6	31
Row 7 →	$\binom{6}{0}\binom{6}{1}\binom{6}{2}\binom{6}{3}\binom{6}{4}\binom{6}{5}\binom{6}{6}$	1 6 15 20 15 6 1	7	57
Row 8 →	$\binom{7}{0}\binom{7}{1}\binom{7}{2}\binom{7}{3}\binom{7}{4}\binom{7}{5}\binom{7}{6}\binom{7}{7}$	1 7 21 35 35 21 7 1	8	99

Fig. 27. Comparing the original rows of Pascal's Triangle to the rows with combinations.

In Figure 27, the rows of combinations are equal to the original rows of Pascal's Triangle, which explains why we are able to do this. In addition, looking at the table, the number of points is equal to the row of Pascal's Triangle being used. We are going to see if we can understand the connection between Pascal's Triangle and the number of regions by adding the first five combinations in the fifth and sixth rows of Pascal's Triangle and comparing it to the value from using the combination formula for R. Let's try it using a circle with six points!

A circle with six points uses the first five terms of the sixth row of Pascal's Triangle. We can rewrite the first five combinations in the sixth row using $\frac{n!}{(n-r)!r!}$ to get

$$\frac{5!}{5!\,(5-5)!}+\frac{5!}{4!\,(5-4)!}+\frac{5!}{3!\,(5-3)!}+\frac{5!}{2!\,(5-2)!}+\frac{5!}{1!\,(5-1)!}.$$

Simplify the denominators.

$$\frac{5!}{5!\,(0!)}+\frac{5!}{4!\,(1!)}+\frac{5!}{3!\,(2!)}+\frac{5!}{2!\,(3!)}+\frac{5!}{1!\,(4!)}$$

Simplify.

$$\frac{5*4*3*2*1}{5*4*3*2*1*1}+\frac{5*4*3*2*1}{4*3*2*1*1}+\frac{5*4*3*2*1}{3*2*1*2*1}+\frac{5*4*3*2*1}{2*1*3*2*1}+\frac{5*4*3*2*1}{1*4*3*2*1}$$

Cancel out the numbers in the numerators and denominators that are in common.

$$\frac{\cancel{5*4*3*2*1}}{\cancel{5*4*3*2*1*}1}+\frac{5*\cancel{4*3*2*1}}{\cancel{4*3*2*1*}1}+\frac{5*4\cancel{*3*2*1}}{\cancel{3*2*1*}2*1}+\frac{5*4\cancel{*3*2*1}}{2*1\cancel{*3*2*1}}+\frac{5\cancel{*4*3*2*1}}{1\cancel{*4*3*2*1}}$$

Simplify.

$$\frac{1}{1}+\frac{5}{1}+\frac{5*4}{2*1}+\frac{5*4}{2*1}+\frac{5}{1}=1+5+10+10+5=31$$

From adding the first five combinations in the sixth row, we get 31 regions, which is true for a circle with six points. Let's see what happens when we use the combination formula for R.

Substitute $\binom{6}{4}$, $\binom{6}{2}$ and $\binom{6}{0}$ into the combination formula for R since $n=6$.

$$R=\binom{6}{4}+\binom{6}{2}+\binom{6}{0}$$

Rewrite the combinations using $\frac{n!}{(n-r)!r!}$.

$$R=\frac{6!}{4!\,(6-4)!}+\frac{6!}{2!\,(6-2)!}+\frac{6!}{0!\,(6-0)!}$$

Simplify the denominators of each fraction.

$$R=\frac{6!}{4!\,2!}+\frac{6!}{2!\,4!}+\frac{6!}{0!\,6!}$$

Simplify.

$$R=\frac{6*5*4*3*2*1}{4*3*2*1*2*1}+\frac{6*5*4*3*2*1}{2*1*4*3*2*1}+\frac{6*5*4*3*2*1}{1*6*5*4*3*2*1}$$

Cancel out the numbers in the numerator and denominator that are in common.

$$R=\frac{6*5\cancel{*4*3*2*1}}{\cancel{4*3*2*1*}2*1}+\frac{6*5\cancel{*4*3*2*1}}{2*1\cancel{*4*3*2*1}}+\frac{\cancel{6*5*4*3*2*1}}{1\cancel{*6*5*4*3*2*1}}$$

Simplify.

$$R = \frac{6*5}{2*1} + \frac{6*5}{2*1} + \frac{1}{1} = 15 + 15 + 1 = 31$$

From this, we can see how the sum of the first five entries equals the value obtained from the combination formula for R. This can also be shown with a circle that has seven points. When adding the first five combinations in the seventh row of Pascal's Triangle, we get

$$\frac{6!}{6!\,0!} + \frac{6!}{5!\,1!} + \frac{6!}{4!\,2!} + \frac{6!}{3!\,3!} + \frac{6!}{2!\,4!} = 57.$$

For the combination formula for R, after we substitute in $\binom{7}{4}$, $\binom{7}{2}$ and $\binom{7}{0}$ and simplify, we get

$$R = \frac{7!}{4!\,3!} + \frac{7!}{2!\,5!} + \frac{7!}{0!\,7!} = 57.$$

Now why do the first five combinations of a row in Pascal's Triangle equal the number from the combination formula for R? We need to prove this algebraically.

Finding the Relationship Algebraically

We left off attempting to find a correlation between the number of regions and the first five terms of each row of Pascal's Triangle using regression and combinations. We are now going to see if we can algebraically find this relationship. The combination formula is

$$R = \binom{n}{4} + \binom{n}{2} + \binom{n}{0}$$

The first five terms of Pascal's Triangle, using combinations, can be expressed as

$$R = \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}.$$

We can now set these two expressions equal to one another to get

$$\binom{n}{4}+\binom{n}{2}+\binom{n}{0}\overset{?}{=}\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}$$

Since $\binom{n}{0}$ and $\binom{n-1}{0}$ are equal to 1, we can eliminate them in the equation. This leaves us with

$$\binom{n}{4}+\binom{n}{2}\overset{?}{=}\binom{n-1}{1}+\binom{n-1}{2}+\binom{n-1}{3}+\binom{n-1}{4}$$

Express each combination using the formula, $nCr=\frac{n!}{r!(n-r)!}$.

$$\frac{n!}{4!\,(n-4)!}+\frac{n!}{2!\,(n-2)!}\overset{?}{=}\frac{(n-1)!}{1!\,(n-2)!}+\frac{(n-1)!}{2!\,(n-3)!}+\frac{(n-1)!}{3!\,(n-4)!}+\frac{(n-1)!}{4!\,(n-5)!}$$

Expand $n!$ and $(n-1)!$ to eliminate the factorials in the denominators.

$$\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)}{2!}\overset{?}{=}\frac{(n-1)}{1!}+\frac{(n-1)(n-2)}{2!}+\frac{(n-1)(n-2)(n-3)}{3!}+\frac{(n-1)(n-2)(n-3)(n-4)}{4!}$$

Multiply the right side of the equation by n and $\frac{1}{n}$.

$$\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)}{2!}\overset{?}{=}\frac{1}{n}\left(n\left[\frac{(n-1)}{1!}+\frac{(n-1)(n-2)}{2!}+\frac{(n-1)(n-2)(n-3)}{3!}+\frac{(n-1)(n-2)(n-3)(n-4)}{4!}\right]\right)$$

Distribute n.

$$\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)}{2!}\overset{?}{=}\frac{1}{n}\left(\frac{n(n-1)}{1!}+\frac{n(n-1)(n-2)}{2!}+\frac{n(n-1)(n-2)(n-3)}{3!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}\right)$$

Get a common denominator of 4! on both sides of the equation.

$$\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{4*3*n(n-1)}{4*3*2!}\overset{?}{=}\frac{1}{n}\left(\frac{4*3*2*n(n-1)}{4*3*2*1!}+\frac{4*3*n(n-1)(n-2)}{4*3*2!}+\frac{4*n(n-1)(n-2)(n-3)}{4*3!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}\right)$$

Simplify.

$$\frac{n(n-1)(n-2)(n-3)}{4!}+\frac{12n(n-1)}{4!}\overset{?}{=}\frac{1}{n}\left(\frac{24n(n-1)}{4!}+\frac{12n(n-1)(n-2)}{4!}+\frac{4n(n-1)(n-2)(n-3)}{4!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{4!}\right)$$

Combine the numerators.

$$\frac{n(n-1)(n-2)(n-3)+12n(n-1)}{4!}\overset{?}{=}\frac{1}{n}\left(\frac{24n(n-1)+12n(n-1)(n-2)+4n(n-1)(n-2)(n-3)+n(n-1)(n-2)(n-3)(n-4)}{4!}\right)$$

Multiply both sides by 4!.

$$n(n-1)(n-2)(n-3)+12n(n-1)\overset{?}{=}\frac{1}{n}(24n(n-1)+12n(n-1)(n-2)+4n(n-1)(n-2)(n-3)+n(n-1)(n-2)(n-3)(n-4))$$

The colors are used to show the expressions that will be distributed.

$$n(n-1)(n-2)(n-3)+12n(n-1)\overset{?}{=}\frac{1}{n}(24n(n-1)+12n(n-1)(n-2)+4n(n-1)(n-2)(n-3)+n(n-1)(n-2)(n-3)(n-4))$$

Distribute n over $(n-1)$, $12n$ over $(n-1)$ twice, $24n$ over $(n-1)$, $4n$ over $(n-1)$ and n over $(n-1)$.

$$(n^2-n)(n-2)(n-3)+12n^2-12n\overset{?}{=}\frac{1}{n}(24n^2-24n+(12n^2-12n)(n-2)+(4n^2-4n)(n-2)(n-3)+(n^2-n)(n-2)(n-3)(n-4))$$

Distribute (n^2-n) over $(n-2)$, $(12n^2-12n)$ over $(n-2)$, $(4n^2-4n)$ over $(n-2)$ and (n^2-n) over $(n-2)$.

$$(n^3-3n^2-2n)(n-3)+12n^2-12n\overset{?}{=}\frac{1}{n}(24n^2-24n+12n^3-36n^2+24n+(4n^3-12n^2+8n)(n-3)+(n^3-3n^2-2n)(n-3)(n-4))$$

Distribute (n^3-2n^2-2n) over $(n-3)$, $(4n^3-12n^2+8n)$ over $(n-3)$ and (n^3-2n^2-2n) over $(n-3)$.

$$n^4-6n^3+11n^2-6n+12n^2-12n\overset{?}{=}\frac{1}{n}(24n^2-24n+12n^3-36n^2+24n+4n^4-24n^3+44n^2-24n+(n^4-6n^3+11n^2-6n)(n-4))$$

On the right side of the equation, distribute $(n^4-6n^3+11n^2-6n)$ over $(n-4)$.

$$n^4-6n^3+11n^2-6n+12n^2-12n\overset{?}{=}\frac{1}{n}(24n^2-24n+12n^3-36n^2+24n+4n^4-24n^3+44n^2-24n+n^5-10n^4+35n^3-50n^2+24n)$$

Combine the colorized like terms.

$$n^4 - 6n^3 + 23n^2 - 18n \stackrel{?}{=} \frac{1}{n}(n^5 - 6n^4 + 24n^3 - 18n^2)$$

Distribute $\frac{1}{n}$ to the right side of the equation.

$$n^4 - 6n^3 + 23n^2 - 18n = n^4 - 6n^3 + 23n^2 - 18n$$

As we can see, we have proved that the number of regions created by n points and the sum of the first five terms of the $(n+1)^{\text{st}}$ row of Pascal's Triangle are equal algebraically!

RECOMMENDATIONS FOR FURTHER RESEARCH

Throughout this paper, we have been finding the maximum number of regions, or the greatest number of regions, in circles with 1-8 points. We can extend this to the *minimum regions* in a circle by making three of the chords intersect to take away some regions. Let's try this with circles that have four and five points! This attempt is shown in Figure 28.

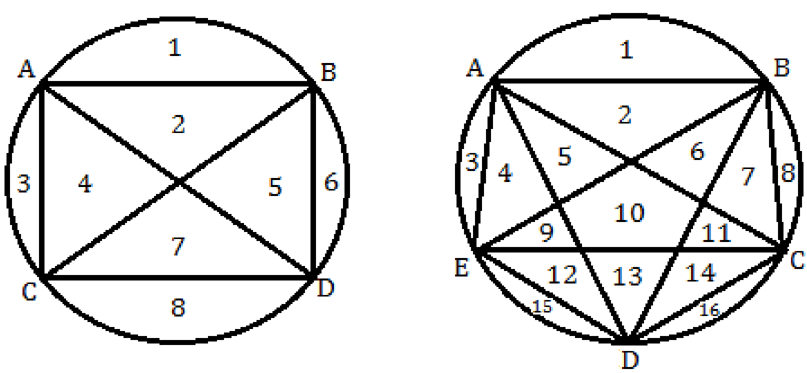

Fig. 28. For circles with four and five points, the minimum equals the maximum.

Thus, we cannot make three or more chords intersect with these circles. But what about a circle with six points? This is shown in Figure 29.

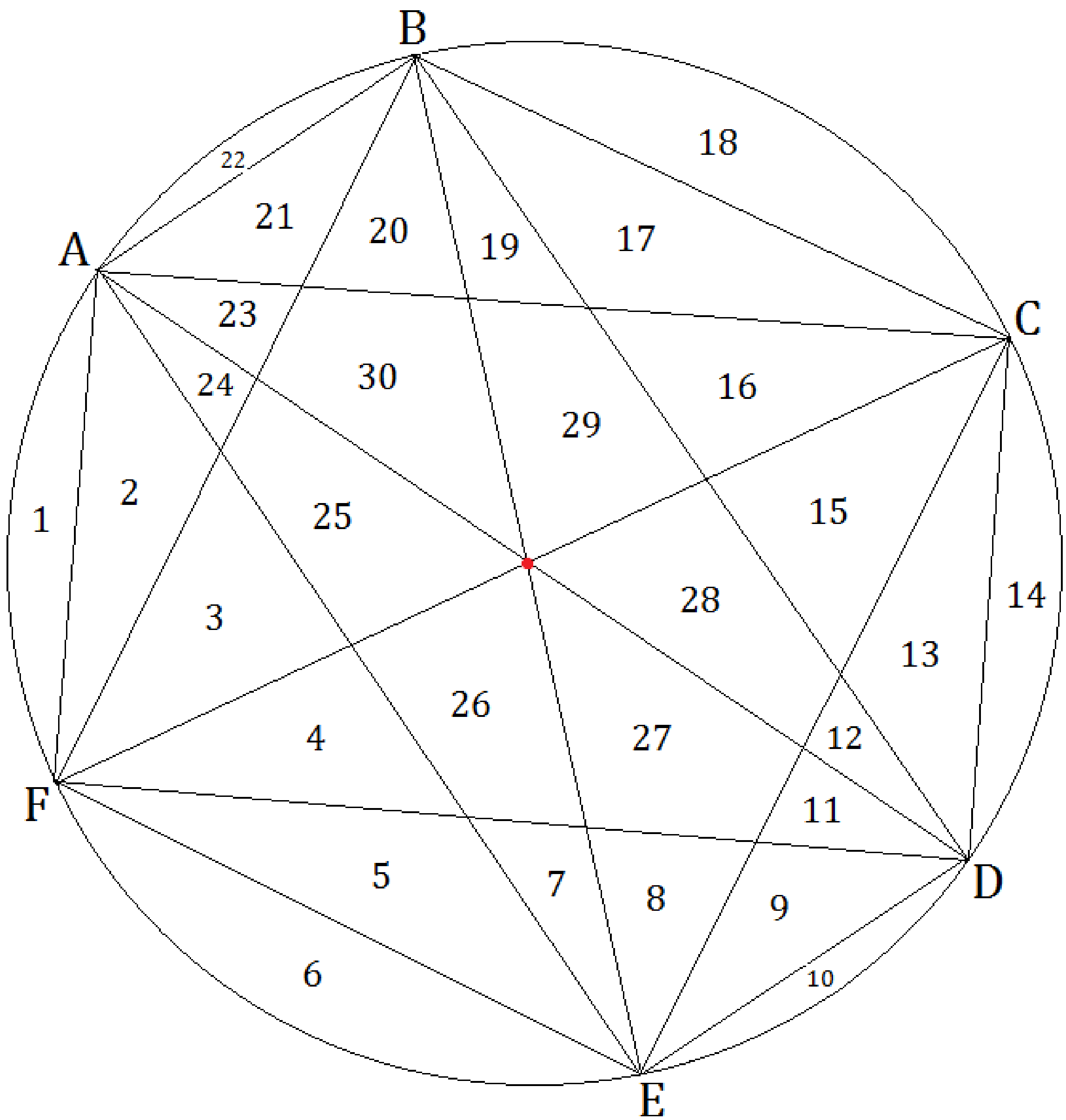

Fig. 29. There are only 30 regions, compared to 31 regions in the other circle with six points.

Further research should include this extension, and possibly come up with a formula to compute the minimum number of regions for a circle with any number of points.

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