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# AN AREA PARADOX CREATED BY THE DISSECTION OF A SQUARE AND THE REARRANGEMENT OF THE POLYGONS CREATED BY THE DISSECTION

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ABSTRACT

This paper explores an area paradox which involves the dissection of a square into two congruent triangles and two congruent trapezoids. The trapezoids and triangles are then rearranged into a rectangle. When rearranging the polygons, there is a difference in the area of the rectangle and square. This paper discusses how Fibonacci numbers and slope can be used to explain this discrepancy. The paper then goes into the process of testing an original sequence.

## PROBLEM STATEMENT

The **area** of a plane figure is the amount of space inside the borders of a 2-dimensional object such as a triangle or a square.

The area of a triangle formula is

$$A = \frac{1}{2}bh.$$

The area of a square formula is

$$A = bh.$$

If we want to find the area of a square with a base of 8, we would multiply 8 x 8 as shown below.

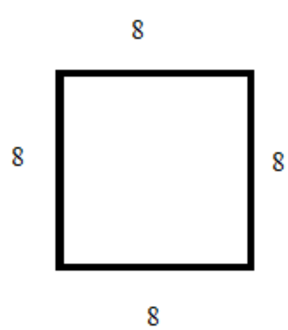



Figure 1. The base of the square is 8, which is one of the Fibonacci numbers.

The area of the square is 64.

$$A = (8)(8) \qquad A = 64$$

Let's say that we want to rearrange this square into a triangle by cutting the square diagonally across and placing the two red sides of them together.

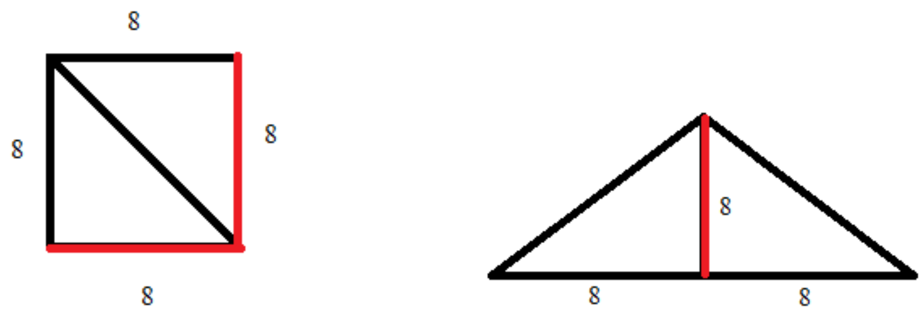



Figure 2. The sides of the square highlighted in red are put together to form the triangle.

The base of the triangle is 16, and the height is 8. Therefore, the area of the triangle is 64.

$$A = \frac{1}{2}(16)(8) \qquad A = 64$$

Both the triangle and the rectangle have the same area, 64.

This time, we will cut the square vertically in the center of the square.

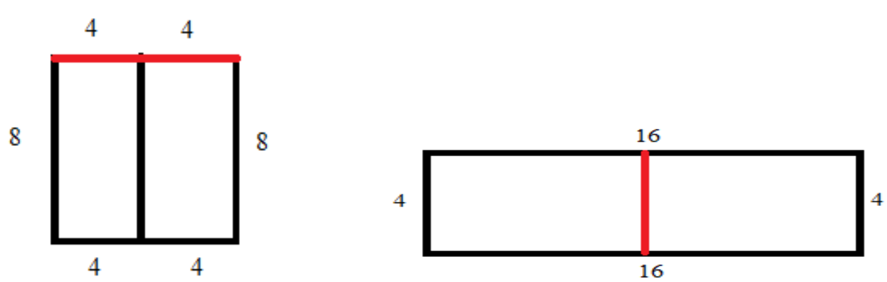



Figure 3. The sides of the square highlighted in red are put together to form the rectangle.

The base of the rectangle is 16, and the height is 4; therefore, the area of the triangle is 64.

$$A = (16)(4) \qquad A = 64$$

Both the rectangle and the square have the same area.

Now let's try dividing the square into two congruent right triangles and two congruent trapezoids. Notice how the numbers that are used to make the bases of the triangles and trapezoids are Fibonacci numbers.

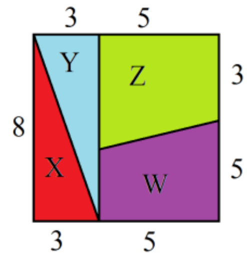



Figure 4. The triangles and trapezoids fit together in order to create the 8x8 square.

The **Fibonacci Sequence** is the endless sequence 1, 1, 2, 3, 5, 8, … in which each term is found by adding the two prior terms. We use the Fibonacci numbers to label our figures because the side broken into two lengths must sum to the side of the square. For example, the figure above is an 8x8 square. Its height is 8 and its base is the sum of 3 and 5 which is 8. If the two numbers added to a number other than 8, then the figure wouldn't be a square.

The area of the square is 64.

$$A = (3 + 5)(8) \qquad A = 64$$

Let's try to rearrange the square into a rectangle.

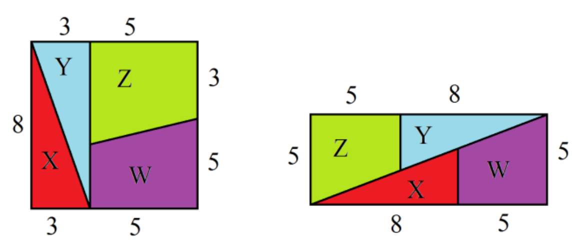



Figure 5. Colors and letters in the square match the corresponding polygon in the rectangle.

The base of the rectangle is 13 and the height of the rectangle is 5.

$$A = (13)(5) \quad A = 65$$

The area of the rectangle is 65.

Even though we used the same polygons that created the square with an area of 64, the area of the rectangle has gotten bigger!

Throughout this paper, we will be discussing how this change in area occurs and why Fibonacci numbers are so important.

- What is the relationship between the Fibonacci numbers and the change in area?
- How is the area of the square changing when rearranged into a rectangle?

## RELATED RESEARCH

We will start by looking at fake images. Fake images are a product of entering numbers into a computer. The values written in the fake image aren't necessarily accurate. We can draw anything, but that doesn't mean it can happen. In the figures below, some of the values written are not possible. The first image in each figure is the fake, and the second image is what the real figure would actually look like.


Figure 6. The circle's radius is 15 while each side of the inscribed square is 3.

The radius of the circle is too large, and therefore the square would not be able to touch the edges of the circle. In the real image, the square would be in the center of the larger circle, without having any contact with the actual circle itself.

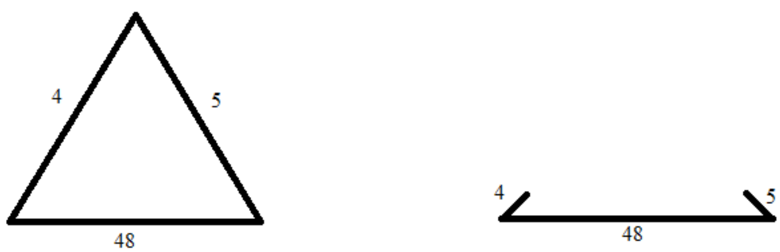



Figure 7. The sides of the "triangle" are 4, 5 and 48.

The sum of any two sides of a triangle must be greater than the third side of the triangle. The lengths of 2 sides of the triangle in the figure above are 4 and 5. The sum of 4 and 5 is less than the third side of the triangle with a length of 48. This means that the sides of the triangle with lengths of 4 and 5 would not be long enough to meet at the top.

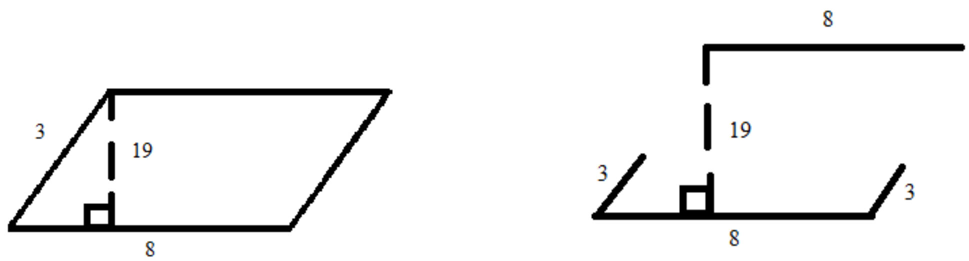



Figure 8. The sides of the parallelogram are 8 and 3 while the height is 19.

The sides of the parallelogram are 3 and 8. Since the height of the parallelogram is 19, the shape must be a fake image. The height of the parallelogram is too large for the shape to be real. In the real shape, the top of the parallelogram would not connect to the rest of the shape.

Now we will start looking at drawn graphs. The purpose of these drawn graphs is to show that the puzzle is real and not just a fake image.

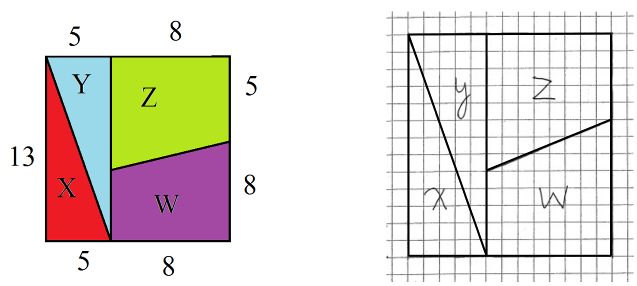



Figure 9. The square on the left is drawn on the grid to the right as per the labeled values.

As seen by the drawn graph, the image to the left can be copied to the graph with the exact values written. Therefore, 5, 8, and 13 are valid numbers to label the square. With this in mind, the area of the square is 169.

$$A = (5 + 8)(13) \qquad A = 169$$

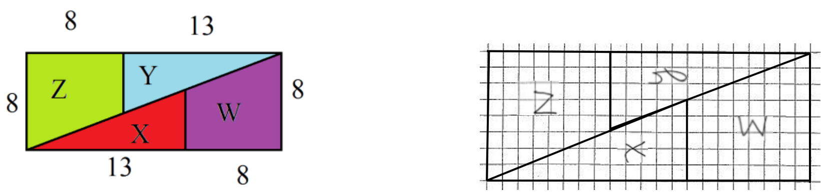



Figure 10. The labeled values in the rectangle on the left are drawn on the grid to the right.

The polygons that made the square could be rearranged to form a rectangle, as seen by the image on the left. We then rearranged the drawn graph of the square into the rectangle on the right. The values of the drawn graph match the values written on the image on the left. This proves that this puzzle is not a fake image and that it is really possible; however, the area of the rectangle is 168.

$$A = (13 + 8)(8) \qquad A = 168$$

In the previous cases, the written values would not work if the images were drawn to scale on a graph; however, the square and rectangle above clearly show that these values are valid. This makes this problem even more puzzling because it can actually work!

Now let's see if other numbers can be substituted in without making the problem a fake image. We can try using other Fibonacci numbers to recreate the square and rectangle. The first 8 terms in the Fibonacci sequence are listed in the table below.

Table 1. Fibonacci Numbers

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … |
|---|---|---|---|---|---|---|---|---|---|
| $f_i$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | … |

Let's use 3, 5, and 8 to make the square and the rectangle.

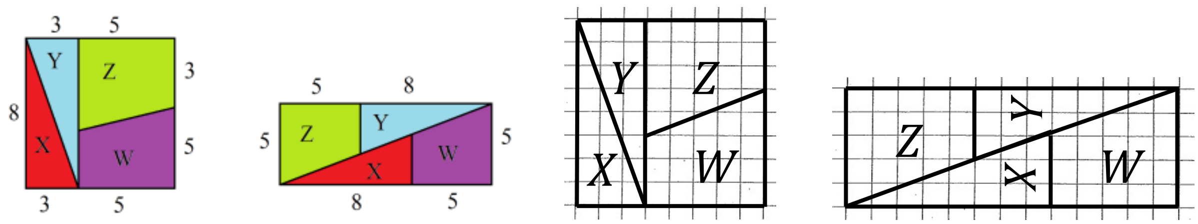



Figure 11. The values in the quadrilaterals to the left are drawn on the grid to the right.

As seen by the drawn graph, the images to the left can be copied to the graph with the exact values written. We then rearranged the square into the rectangle as seen by the figure above. The values of the drawn graph match the values written on the images to the left. Therefore, 3, 5, and 8 are valid numbers to label the square and rectangle. With this in mind, the area of the square is 169.

$$A = (3 + 5)(8) \qquad A = 64$$

And the area of the rectangle is 65.

$$A = (5 + 8)(5) \qquad A = 65$$

Yet again this difference in area appears.

Now let's use 1, 1, and 2.

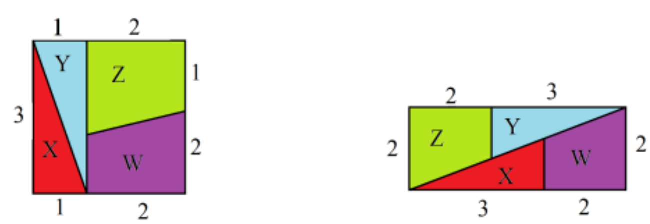



Figure 12. The first 3 Fibonacci numbers are used to label these figures.

These Fibonacci numbers also work to recreate the square and the rectangle. The square is 3x3, and when rearranged, it creates a 2x5 rectangle. The area of the square is 9.

$$A = (1 + 2)(3) \qquad A = 9$$

And the area of the rectangle is 10.

$$A = (2 + 3)(2) \qquad A = 10$$

The same inconsistency found in the area of the original square and rectangle we worked with can be found with the new ones we just created.

Let's try 2, 3, and 5.


Figure 13. The 3rd, 4th, and 5th Fibonacci numbers are used to label the figures.

These Fibonacci numbers also work to recreate the square and the rectangle. The square is 5x5, and when rearranged, it creates a 3x8 rectangle. The area of the square is 25.

$$A = (2 + 3)(5) \qquad A = 25$$

And the area of the rectangle is 24.

$$A = (3 + 5)(3) \qquad A = 24$$

Once again this difference in area appears.

Now let's try 8, 13, and 21.

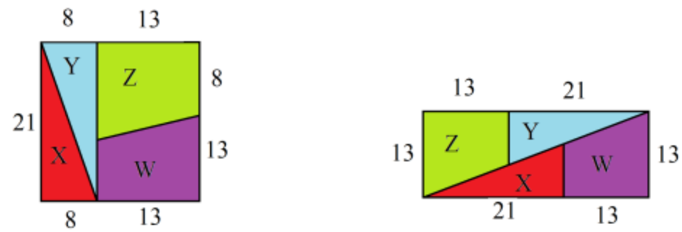



Figure 14. The 6th, 7th, and 8th Fibonacci numbers are used to label the figures.

These Fibonacci numbers also work to recreate the square and the rectangle. The square is 21x21, and when rearranged, it creates a 13x34 rectangle. With this in mind, the area of the square is 441.

$$A = (8 + 13)(21) \qquad A = 441$$

And the area of the rectangle is 442.

$$A = (13 + 21)(13) \qquad A = 442$$

The same inconsistency found in the area of the other squares and rectangles we worked with can be found with the new ones we just created.

Let's try 13, 21, and 34.

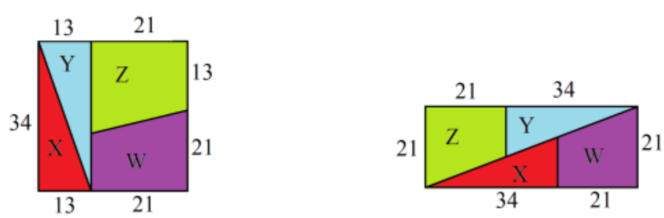



Figure 15. The 7th, 8th, and 9th Fibonacci numbers are used to label the figures.

These Fibonacci numbers also work to recreate the square and the rectangle. The square is 34x34, and when rearranged, it creates a 21x55 rectangle. With this in mind, the area of the square is 1156.

$$A = (13 + 21)(34) \qquad A = 1156$$

And the area of the rectangle is 1155.

$$A = (21 + 34)(21) \qquad A = 1155$$

Yet again this difference in area appears.

Each set of Fibonacci numbers had a change in the area when rearranging the square. As you can see by the table below, a pattern forms when you subtract the area of the square from the area of the rectangle. The differences alternate between 1 and -1.

Table 2. Differences of the Areas

| 3 Fibonacci Numbers | Area of the Square | Area of the Rectangle | Area of the Rectangle – the Area of the Square |
|---|---|---|---|
| 1,2,3 | 9 | 10 | 1 |
| 2,3,5 | 25 | 24 | -1 |
| 3,5,8 | 64 | 65 | 1 |
| 5,8,13 | 169 | 168 | -1 |
| 8,13,21 | 441 | 442 | 1 |
| 13,21,34 | 1156 | 1155 | -1 |
| … | … | … | … |

Now that we realize the discrepancy, we can start to explore why it happens.

Solving For the Change in Area

Let's look at the rectangle drawn on the axes in Figure 16.

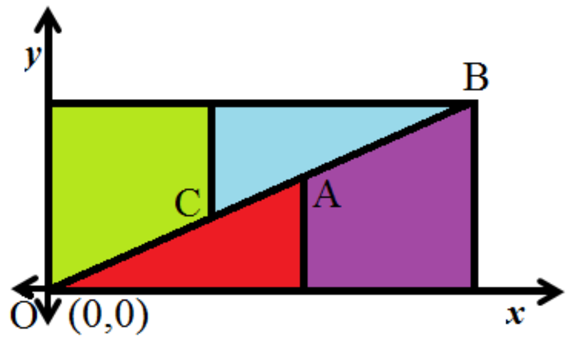



Figure 16. The main points of the figure are labeled O, C, A, and B.

We will use the numbers 2, 3 and 5 as the three Fibonacci numbers. If we label the Figure 16, the rectangle would be marked as shown in Figure 17.

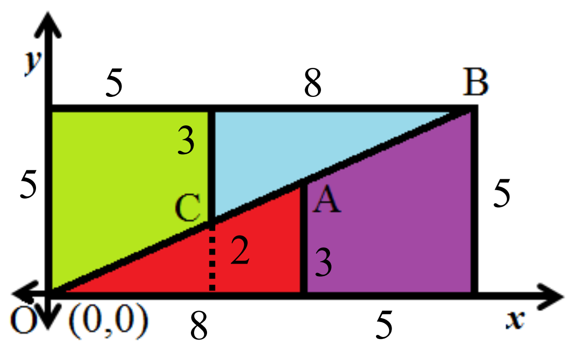



Figure 17. Notice that all the given side lengths are Fibonacci numbers.

Each of the points has its own coordinate on the axes. ***O*** is (0,0), ***C*** is (5,2), ***A*** is (8,3), and ***B*** is (13,5). Using these coordinates, let's find the slope of ***OA*** and ***AB***.

$$\text{slope of } \boldsymbol{OA} = \frac{\mathbf{3}}{\mathbf{8}} \text{ or } \mathbf{0.375} \qquad \text{slope of } \boldsymbol{AB} = \frac{5-3}{13-8} = \frac{\mathbf{2}}{\mathbf{5}} \text{ or } \mathbf{0.4}$$

The slopes are not equal! That means that ***OAB*** is not a straight line! ***AB*** has a steeper slope than ***OA*** which creates a "kinked" line. In this case, the "kinked" line has a "dip" because the slope of ***AB*** is greater than the slope of ***OA***. This causes the difference in area that we discovered previously to be a gap. The polygon created by the gap is a parallelogram. The parallelogram has an area of 1, hence, the rectangle has an area that is greater by 1.

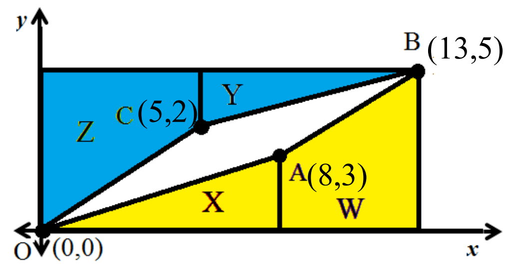



Figure 18. The polygons don't fit perfectly together, and in this case, create a gap.

In other cases, the "kinked" line creates a "hill" because the slope of the first segment is greater than the second. This causes the difference in area to decrease by 1. The polygon created by the overlap is a parallelogram. The parallelogram has an area of 1, hence, the rectangle has an area that is smaller by 1.

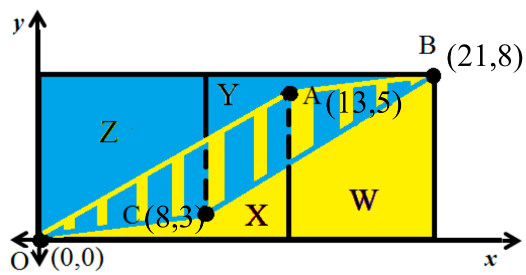
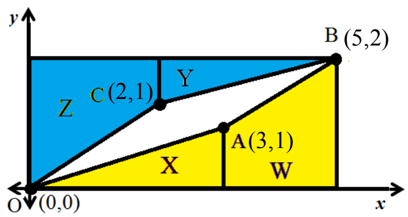
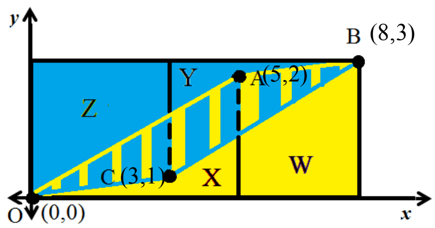
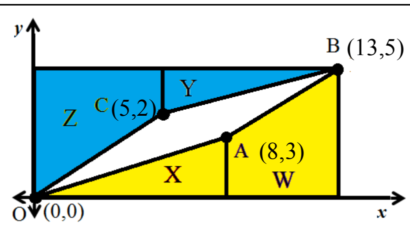
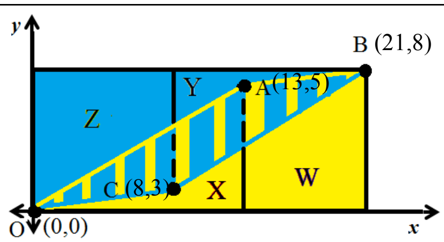
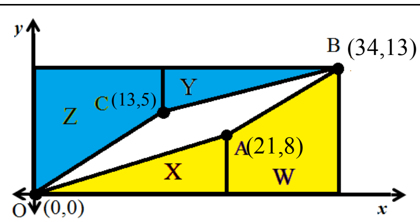
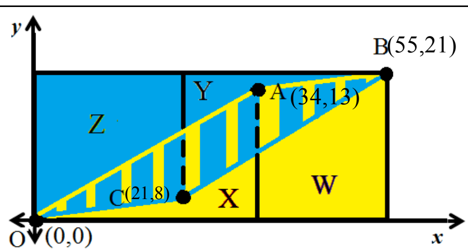



Figure 19. The polygons don't fit perfectly together, and in this case, create an overlap.

If ***OAB*** is not a straight line, then how come we did not notice it when we drew our graphs? Since the slopes are so close to one another, it is almost impossible to see the difference with a naked eye. Table 3 shows the exaggerated difference in slope so we can see the discrepancy visually.

Table 3. Exaggerated Difference in Area Figures

| Exaggerated Figure | 3 Fibonacci Numbers | Slope of $\overline{OA}$ | Slope of $\overline{AB}$ | Gap or Overlap? |
|---|---|---|---|---|
|   | 1,2,3 | $.\overline{3}$ | $.5$ | Gap |
|   | 2,3,5 | $.4$ | $.\overline{3}$ | Overlap |
|   | 3,5,8 | $.375$ | $.4$ | Gap |
|   | 5,8,13 | $.\overline{384615}$ | $.375$ | Overlap |
|   | 8,13,21 | $.\overline{380952}$ | $.\overline{384615}$ | Gap |
|   | 13,21,34 | $.382352941176471$ | $.\overline{380952}$ | Overlap |

Let's list the first ten Fibonacci numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

We can use these numbers to create ratios. The ratios have a Fibonacci number as the numerator and that Fibonacci number plus 2 as the denominator.

$$\frac{f_n}{f_{n+2}}$$

Now let's write ratios for a couple of Fibonacci numbers.

Table 4. Fibonacci Number Ratios

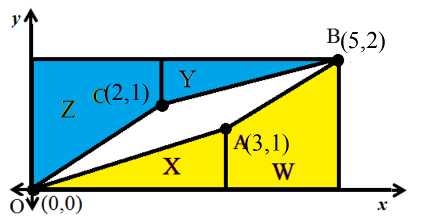
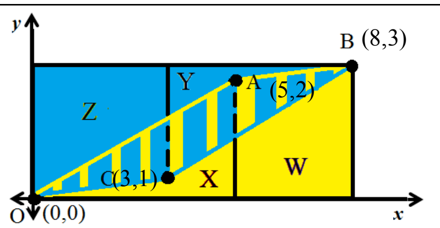
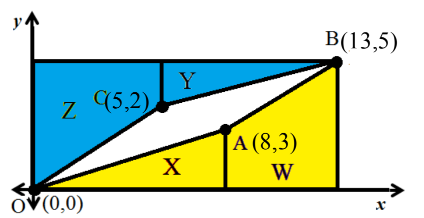
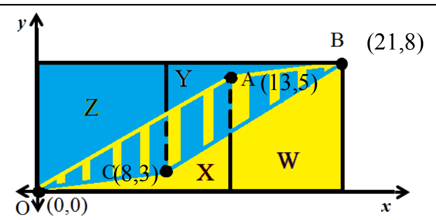
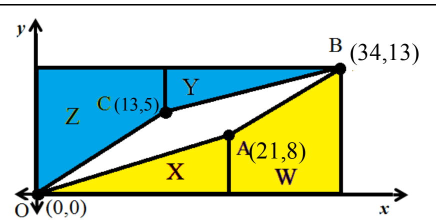
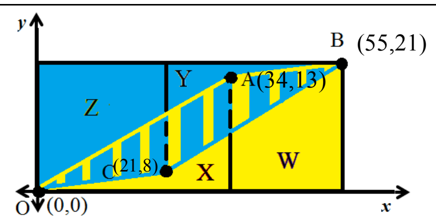
| | | |
|---|---|---|
| $\frac{f_1}{f_3}$ | $\frac{1}{2}$ | $.5$ |
| $\frac{f_2}{f_4}$ | $\frac{1}{3}$ | $.\overline{3}$ |
| $\frac{f_3}{f_5}$ | $\frac{2}{5}$ | $.4$ |
| $\frac{f_4}{f_6}$ | $\frac{3}{8}$ | $.375$ |
| $\frac{f_5}{f_7}$ | $\frac{5}{13}$ | $.\overline{384615}$ |
| $\frac{f_6}{f_8}$ | $\frac{8}{21}$ | $.\overline{380952}$ |
| $\frac{f_7}{f_9}$ | $\frac{13}{34}$ | $.382352941176471$ |
| $\frac{f_8}{f_{10}}$ | $\frac{21}{55}$ | $.3\overline{81}$ |
| … | … | … |
| $\frac{f_{88}}{f_{90}}$ | $\frac{1100087778366101931}{2880067194370816120}$ | 0.381966011250105151795413165634 36 |

Notice how as the Fibonacci numbers increase, the ratio's decimal value approaches 0.382.

$$\text{As } n \to \infty, \ \frac{f_n}{f_{n+2}} \to \approx .382$$

If we look at the slopes of the exaggerated figures closely, we can see that the slopes are equivalent to the ratios we have just created. Table 5 shows the equivalent slopes and ratios.

<u>Table 5. Exaggerated Difference in Area Figures</u>

| Exaggerated Figure | 3 Fibonacci Numbers | Slope of $\overline{OA}$ | Slope of $\overline{AB}$ | Gap or Overlap? |
|---|---|---|---|---|
|   | 1,2,3 | $\frac{f_2}{f_4} = .\overline{3}$ | $\frac{f_1}{f_3} = .5$ | Gap |
|   | 2,3,5 | $\frac{f_3}{f_5} = .4$ | $\frac{f_2}{f_4} = .\overline{3}$ | Overlap |
|   | 3,5,8 | $\frac{f_4}{f_6} = .375$ | $\frac{f_3}{f_5} = .4$ | Gap |
|   | 5,8,13 | $\frac{f_5}{f_7} = .\overline{384615}$ | $\frac{f_4}{f_6} = .375$ | Overlap |
|   | 8,13,21 | $\frac{f_6}{f_8} = .\overline{380952}$ | $\frac{f_5}{f_7} = .\overline{384615}$ | Gap |
|   | 13,21,34 | $\frac{f_7}{f_9} = .382352941176471$ | $\frac{f_6}{f_8} = .\overline{380952}$ | Overlap |

Now that we have tested about 6 cases of this discrepancy, we can begin an algebraic proof to see if it works every time.

<u>Creating An Algebraic Expression</u>

Let's look at the square in Figure 20.

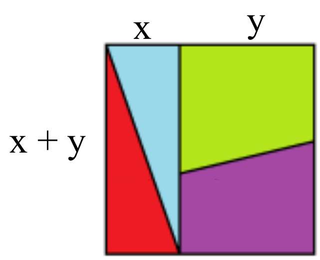



Figure 20. Variables x, y, and x+y represent Fibonacci numbers with which we label the square.

We can label the square with variables so that we can represent any square. Instead of the set of three Fibonacci numbers we use to label the square, x would be equal to the smallest number, y would be equal to the middle number, and x + y would be equal to the largest number.

Keeping this in mind, the area of any of the squares, A, would equal $(x+y)^2$.

$$A = (x+y)(x+y)$$

Multiply.

$$A = (x+y)^2$$

The rectangle in Figure 21 is labeled with the variables that correspond with the measures of the rectangle.

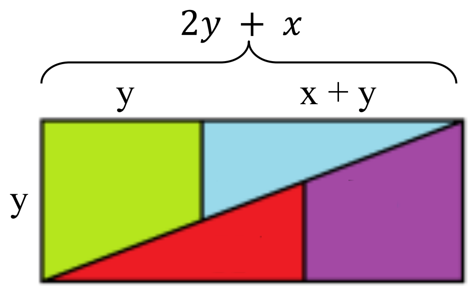



Figure 21. Variables x, y, and x+y represent the Fibonacci numbers we label the rectangle with.

The area of any of the rectangles, A, would equal $2y^2 + xy$.

$$A = (y)(2y + x)$$

Distribute.

$$A = 2y^2 + xy$$

The difference in area of any square and rectangle pair, D, is

$$D = (2y^2 + xy) - (x + y)^2.$$

Square the binomial.

$$D = (2y^2 + xy) - (x^2 + 2xy + y^2)$$

Distribute the -1.

$$D = 2y^2 + xy - x^2 - 2xy - y^2$$

Combine like terms.

$$D = y^2 - xy - x^2$$

The difference in area equation should be equal to 0.

$$y^2 - xy - x^2 = 0$$

However, it has been equal to $\pm 1$ because of the slope difference.

$$y^2 - xy - x^2 = \pm 1$$

Now let's put this expression into Fibonacci notation.

$$y^2 - xy - x^2$$

Factor out $-x$.

$$y^2 - x(y + x)$$

Since x, y, and x + y are consecutive Fibonacci numbers, substitute where x = $f_i$, y = $f_{i+1}$, and x + y = $f_{i+2}$.

$$f_{i+1}{}^2 - f_i(f_{i+2})$$

We can try this expression with the first ten sets of three Fibonacci numbers.

Table 6. Differences in Area Expressions in Fibonacci Notation

| $f_i$ | $f_{i+1}$ | $f_{i+2}$ | $f_{i+1}{}^2 - f_i(f_{i+2})$ |
|---|---|---|---|
| 1 | 1 | 2 | $1^2 - 1(2) = -1$ |
| 1 | 2 | 3 | $2^2 - 1(3) = 1$ |
| 2 | 3 | 5 | $3^2 - 2(5) = -1$ |
| 3 | 5 | 8 | $5^2 - 3(8) = 1$ |
| 5 | 8 | 13 | $8^2 - 5(13) = -1$ |
| 8 | 13 | 21 | $13^2 - 8(21) = 1$ |
| 13 | 21 | 34 | $21^2 - 13(34) = -1$ |
| 21 | 34 | 55 | $34^2 - 21(55) = 1$ |
| 34 | 55 | 89 | $55^2 - 34(89) = -1$ |
| 55 | 89 | 144 | $89^2 - 55(144) = 1$ |

Notice how all of the sets of Fibonacci numbers make the expression come out to be $\pm 1$.

Let's try the same equation on random sets of three Fibonacci numbers.

Table 7. Differences in Area Expressions in Fibonacci Notation

| $f_i$ | $f_{i+1}$ | $f_{i+2}$ | ${f_{i+1}}^2 - f_i(f_{i+2})$ |
|---|---|---|---|
| 102334155 | 165580141 | 267914296 | $165580141^2 - 102334155(267914296) = 1$ |
| 190392490709135 | 308061521170129 | 498454011879264 | $308061521170129^2 - 190392490709135(498454011879264) = 1$ |
| 1100087778366101931 | 1779979416004714189 | 2880067194370816120 | $1779979416004714189^2 - 1100087778366101931(2880067194370816120) = 1$ |

All of the sets of Fibonacci numbers tested still make the expression come out to be $\pm 1$.

We know that when using a set of three consecutive Fibonacci numbers we can conjecture the middle number squared minus the product of smallest number and the largest number; however, we must prove that expression equals $\pm 1$.

## An Algebraic Proof that ${f_{i+1}}^2 - f_i(f_{i+2}) = \pm 1$

Before doing the proof, we must understand the following fact. Using the following formula, we can find every Fibonacci number.

$$f_x = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x}{\sqrt{5}}$$

We can put this formula in $Y_1$ and look at the table.

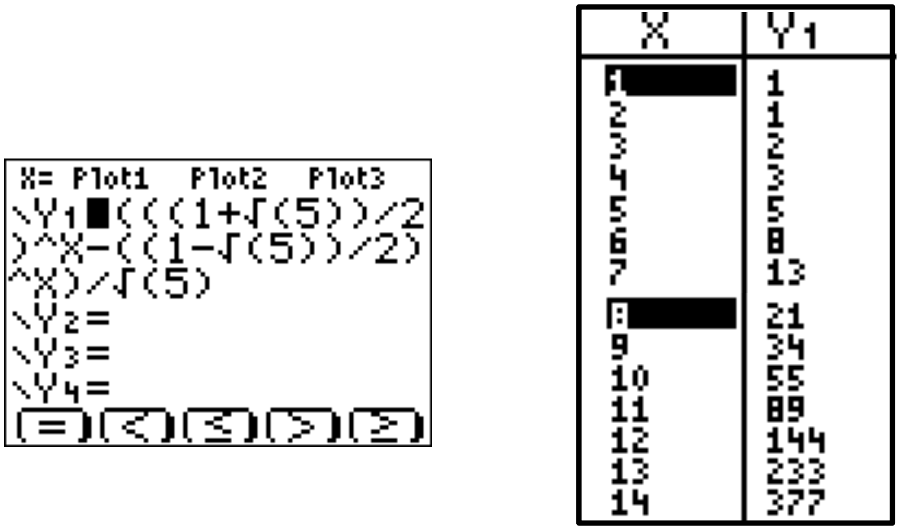



Figure 22. All of the numbers for each value of $x$ are Fibonacci numbers.

Now that we know the fact is true, we can use it and some algebra to prove that ${f_{i+1}}^2 - f_i(f_{i+2})$ is equal to $\pm 1$.

Since that formula is equivalent to any Fibonacci number, the smallest of our three consecutive Fibonacci numbers, $f_i$, would be the original formula.

$$f_x = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x}{\sqrt{5}}$$

The middle number, $f_{i+1}$, would be the same formula, but with $x+1$ as the subscript.

$$f_{x+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{x+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{x+1}}{\sqrt{5}}$$

The largest number, $f_{i+2}$, would be the same formula, but with $x+2$ as the subscript.

$$f_{x+2} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{x+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{x+2}}{\sqrt{5}}$$

Now let's substitute these formulas into the expression ${f_{i+1}}^2 - f_i(f_{i+2})$.

$$ {f_{x+1}}^2 - f_x(f_{x+2}) = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{x+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{x+1}}{\sqrt{5}}\right]^2 $$

$$ - \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{x} - \left(\frac{1-\sqrt{5}}{2}\right)^{x}}{\sqrt{5}}\right]\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{x+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{x+2}}{\sqrt{5}}\right] $$

Square and multiply.

$$ = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} - 2\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{x+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2x+2}}{5}\right] $$

$$ - \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} - \left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{x+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{x+2} + \left(\frac{1+\sqrt{5}}{2}\right)^{2x+2}}{5}\right] $$

Distribute the $-1$.

$$= \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} - 2\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{x+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2x+2}}{5}\right]$$

$$+ \left[\frac{-\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} + \left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{x+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{x+2} - \left(\frac{1+\sqrt{5}}{2}\right)^{2x+2}}{5}\right]$$

Highlight the additive inverses.

$$= \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} - 2\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{x+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{2x+2}}{5}\right]$$

$$+ \left[\frac{-\left(\frac{1+\sqrt{5}}{2}\right)^{2x+2} + \left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{x+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{x+2} - \left(\frac{1+\sqrt{5}}{2}\right)^{2x+2}}{5}\right]$$

Combine the fractions. Notice how the expressions highlighted in green and blue cancel out with the expression in the corresponding color, above, to get the equation below. Next, we will separate each of the highlighted expressions, below, into factors.

$$f_{x+1}{}^{2} - f_x(f_{x+2}) = \frac{-2\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{x+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{x+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{x+2}}{5}$$

Factor out the numerators. Notice how the expressions highlighted in pink, yellow and red, above, factor out to get the equation below.

$$f_{x+1}{}^{2} - f_x(f_{x+2})$$

$$= \frac{-2\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{x}\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^{1} + \left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1-\sqrt{5}}{2}\right)^{2} + \left(\frac{1-\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{x}\left(\frac{1+\sqrt{5}}{2}\right)^{2}}{5}$$

Factor out the GCF which is $\frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^x}{5}$.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = \frac{\left[\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)\right]^x}{5}\left[-2\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)+\left(\frac{1-\sqrt{5}}{2}\right)^2+\left(\frac{1+\sqrt{5}}{2}\right)^2\right]$$

Simplify the terms in each set of brackets.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = \frac{\left[\frac{1-\sqrt{5}+\sqrt{5}-5}{4}\right]^x}{5}\left[\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right]^2$$

Simplify. Notice how the numbers highlighted in yellow and blue, above, cancel each other out, below.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = \left[\frac{(-1)^x}{5}\right]\left[\left(\frac{2\sqrt{5}}{2}\right)^2\right]$$

Square the second fraction.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = \left[\frac{(-1)^x}{5}\right]\left[\frac{4(5)}{4}\right]$$

Divide the numerator and denominator of the second fraction by 4.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = \left[\frac{(-1)^x}{5}\right][5]$$

Multiply.

$$f_{x+1}{}^2 - f_x(f_{x+2}) = (-1)^x$$

This means that if the exponent, x, is an even number, the equation will be equal to $+1$, but if x is an odd number, the equation will be equal to $-1$. In other words,

$$f_{x+1}^{\ 2} - f_x(f_{x+2}) = \pm 1.$$

Now that we know that if the exponent, x, is an even number, the equation will be equal to $+1$, and if x is an odd number, the equation will be equal to $-1$, we can guess the difference in area for any set of three Fibonacci numbers.

Table 8. Guessing the Difference in Area

| $f_i$ | $f_{i+1}$ | $f_{i+2}$ | $i$ | **Is $i$ odd or even?** | $\pm 1$ |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 1 | odd | $-1$ |
| 1 | 2 | 3 | 2 | even | 1 |
| 2 | 3 | 5 | 3 | odd | $-1$ |
| 3 | 5 | 8 | 4 | even | 1 |
| 5 | 8 | 13 | 5 | odd | $-1$ |
| 8 | 13 | 21 | 6 | even | 1 |
| 13 | 21 | 34 | 7 | odd | $-1$ |
| 21 | 34 | 55 | 8 | even | 1 |
| 34 | 55 | 89 | 9 | odd | $-1$ |
| 55 | 89 | 144 | 10 | even | 1 |
| … | … | … | … | … | … |

Now that we have explained how this paradox works with three consecutive Fibonacci numbers, we can test an original sequence.

## Testing an Original Sequence

Let’s create our own Fibonacci-like sequence. Each number must be the sum of the two previous terms. We can call it the Leventhal Sequence. The sequence starts where $F_1 = 3$. The next term is $F_2 = 3$. The previous two terms, 3 and 3, sum to the next term, so $F_3 = 6$ and so on and so forth. This creates the following sequence:

3,3,6,9,15,24,39,63,102…

Now that we have an original sequence, we can label a square. Let's start with the first three consecutive Leventhal numbers; 3, 3 and 6.

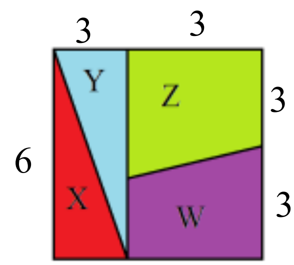



Figure 23. The square is labeled in a similar fashion to the ones with Fibonacci numbers.

The area of the square, A, is 36.

$$A = (3 + 3)(6) \qquad A = 36$$

Let's try to rearrange the square into a rectangle.

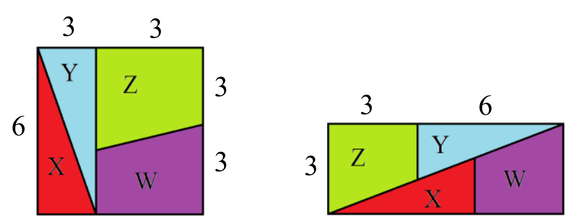



Figure 24. The square is rearranged in the same way as the ones with Fibonacci numbers.

The base of the rectangle is 9 and the height of the rectangle is 3.

$$A = (9)(3) \qquad A = 27$$

The area of the rectangle, A, is 27.

Let’s find the difference in area, D.

$$D = 27 - 36 \quad D = -9$$

The difference in area when the square is subtracted from the rectangle is $-9$.

Let’s test some more to see if a pattern forms. We can use the next three consecutive Leventhal Numbers; 3, 6 and 9.

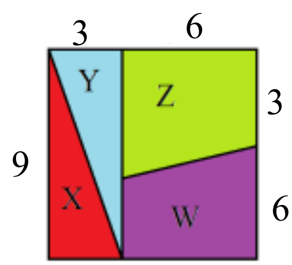



Figure 25. The square is labeled in a similar fashion to the ones with Fibonacci numbers.

The area of the square, A, is 81.

$$A = (3 + 6)(9) \quad A = 81$$

Let’s try to rearrange the square into a rectangle.

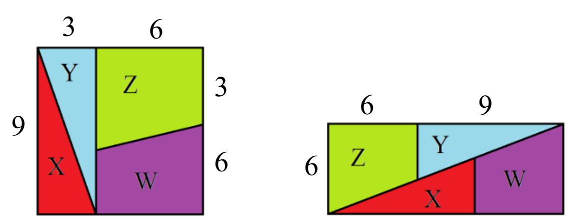



Figure 26. The square is rearranged in the same way as the ones with Fibonacci numbers.

The base of the rectangle is 15 and the height of the rectangle is 6.

$$A = (15)(6) \quad A = 90$$

The area of the rectangle, A, is 90.

Let's find the difference in area, D.

$$D = 90 - 81 \qquad D = 9$$

The difference in area when the square is subtracted from the rectangle is 9.

Let's try another one. We can use the following three consecutive Leventhal Numbers; 6, 9 and 15.

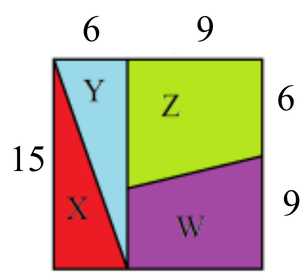



Figure 27. The square is labeled in a similar fashion to the ones with Fibonacci numbers.

The area of the square, A, is 225.

$$A = (6 + 9)(15) \qquad A = 225$$

Let's try to rearrange the square into a rectangle.

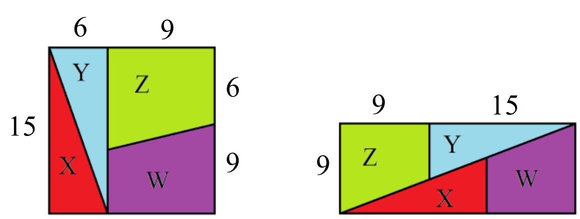



Figure 28. The square is rearranged in the same way as the ones with Fibonacci numbers.

The base of the rectangle is 24 and the height of the rectangle is 9.

$$A = (24)(9) \qquad A = 216$$

The area of the rectangle, A, is 216.

Let's find the difference in area, D.

$$D = 216 - 225 \qquad D = -9$$

The difference in area when the square is subtracted from the rectangle is $-9$.

Let's try another one. We can use the following three consecutive Leventhal Numbers; 9, 15 and 24.

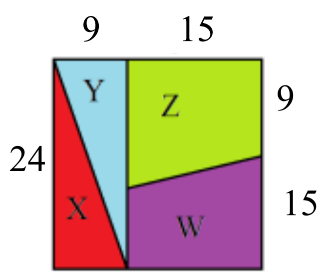



Figure 29. The square is labeled in a similar fashion to the ones with Fibonacci numbers.

The area of the square, A, is 576.

$$A = (9 + 15)(24) \qquad A = 576$$

Let's try to rearrange the square into a rectangle.

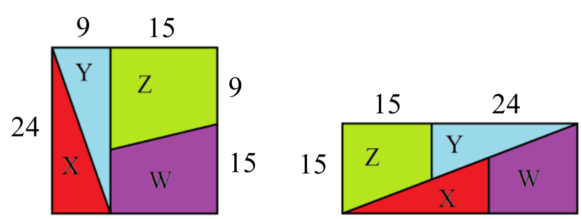



Figure 30. The square is rearranged in the same way as the ones with Fibonacci numbers.

The base of the rectangle is 24 and the height of the rectangle is 9.

$$A = (39)(15) \qquad A = 585$$

The area of the rectangle, A, is 585.

Let's find the difference in area, D.

$$D = 585 - 576 \quad D = 9$$

The difference in area when the square is subtracted from the rectangle is 9.

The pattern that seems to be forming is that the difference in area is $\pm 9$. Let's put this into a table.

Table 9. Differences of the Areas

| Three Leventhal Numbers | Area of the Square | Area of the Rectangle | Area of the Rectangle – the Area of the Square |
|---|---|---|---|
| 3,3,6 | 36 | 27 | -9 |
| 3,6,9 | 81 | 90 | 9 |
| 6,9,15 | 225 | 216 | -9 |
| 9,15,24 | 576 | 585 | 9 |
| 15,24,39 | 1521 | 1512 | -9 |
| 24,39,63 | 3969 | 3978 | 9 |
| 39,63,102 | 10404 | 10395 | -9 |
| 63,102,165 | 27225 | 27234 | 9 |
| … | … | … | … |

There is definitely a difference in area for this paradox using the Leventhal sequence; however, it is much larger than when we used the Fibonacci numbers. The Fibonacci numbers only had a difference of $\pm 1$ while the Leventhal numbers had a difference of $\pm 9$; however, the difference to the area of the square ratio is equal to the Fibonacci numbers.

Table 10. Difference in Area Percentages

| Three Fibonacci Numbers | Absolute Value of the $\frac{\text{Difference}}{\text{Area of the Square}}$ of the Fibonacci Sequence as a % | Three Leventhal Numbers | Absolute Value of the $\frac{\text{Difference}}{\text{Area of the Square}}$ of the Leventhal Sequence as a % |
|---|---|---|---|
| 1,1,2 | $\left\lvert\frac{-1}{4}\right\rvert = 25\%$ | 3,3,6 | $\left\lvert\frac{-9}{36}\right\rvert = 25\%$ |
| 1,2,3 | $\left\lvert\frac{1}{9}\right\rvert = 11.\bar{1}\%$ | 3,6,9 | $\left\lvert\frac{9}{81}\right\rvert = 11.\bar{1}\%$ |
| 2,3,5 | $\left\lvert\frac{-1}{25}\right\rvert = 4\%$ | 6,9,15 | $\left\lvert\frac{-9}{225}\right\rvert = 4\%$ |
| 3,5,8 | $\left\lvert\frac{1}{64}\right\rvert = 1.5625\%$ | 9,15,24 | $\left\lvert\frac{9}{576}\right\rvert = 1.5625\%$ |
| 5,8,13 | $\left\lvert\frac{-1}{169}\right\rvert = 0.591715976331360946745 56\%$ | 15,24,39 | $\left\lvert\frac{-9}{1521}\right\rvert = 0.59171597633136094674556\%$ |
| 8.13.21 | $\left\lvert\frac{1}{441}\right\rvert = 0.22675736961451247165533\%$ | 24,39,63 | $\left\lvert\frac{9}{3969}\right\rvert = 0.22675736961451247165533\%$ |
| 13,21,34 | $\left\lvert\frac{-1}{1156}\right\rvert = 0.08650519031141868512111\%$ | 39,63,102 | $\left\lvert\frac{-9}{10404}\right\rvert = 0.08650519031141868512111\%$ |
| 21,34,55 | $\left\lvert\frac{1}{3025}\right\rvert = 0.03305785123966942148760\%$ | 63,102,165 | $\left\lvert\frac{9}{27225}\right\rvert = 0.03305785123966942148760\%$ |
| … | … | … | … |

Both sequences have equal ratios of difference to area of the square. The paradox works with the same effect using both sequences. Since the difference in area is so small when the numbers that are used approach ∞, it cannot be seen with the naked eye. This makes the paradox truly puzzling.

## Fibonacci Triangle Paradox

We just discussed a paradox using the rearrangement of a square into a rectangle, but does it work with any other polygon? Let's try to use a right triangle.

The area of a right triangle formula is

$$A = \frac{1}{2}bh.$$

Let's look at the right triangle below.

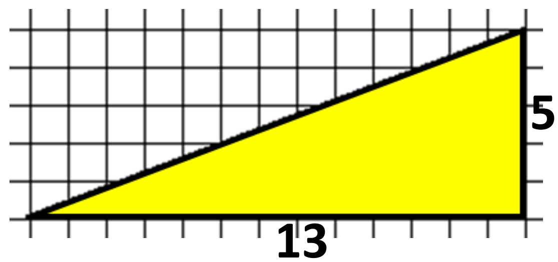



Figure 31. The base of the right triangle is 13 and the height is 5.

The area of the right triangle is 32.5.

$$A = \left(\frac{1}{2}\right)(13)(5) \qquad A = 32.5$$

Now let's split this right triangle into two right triangles and two irregular hexagons as shown in Figure 32.

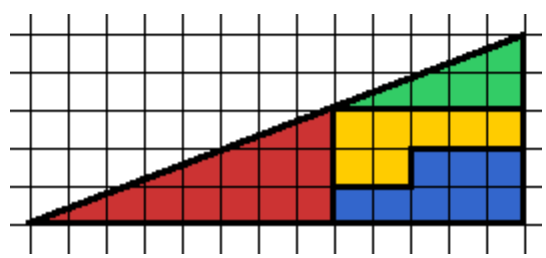


Figure 32. The base of the right triangle is 13 and the height is 5.

Let's find the area of each polygon making up the right triangle shown in Figure 32. The area of the red right triangle is

$$A = \left(\frac{1}{2}\right)(3)(8) \qquad A = 12$$

The area of the green right triangle is

$$A = \left(\frac{1}{2}\right)(2)(5) \qquad\qquad A = 5$$

The area of the blue irregular hexagon is

$$A = 8$$

The area of the yellow irregular hexagon is

$$A = 7$$

The total area of the right triangle when adding the areas of the polygons is

$$A = 12 + 5 + 8 + 7 \qquad\qquad A = 32$$

What's going on?!? The area of the original right triangle was 32.5, but now it is 32, so there must be a reason! Let's try to rearrange the right triangle above into the right triangle shown below.

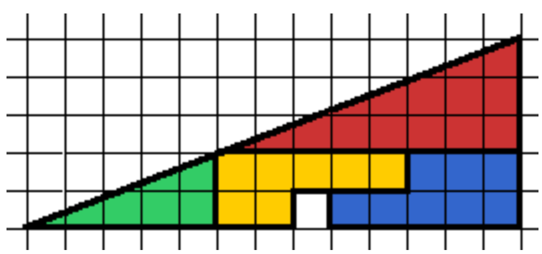


Figure 33. The base of the right triangle is 13 and the height is 5.

The area of the right triangle should be 32.5.

$$A = \left(\frac{1}{2}\right)(13)(5) \qquad\qquad A = 32.5$$

But why is there an extra white square? Let's find the area of each polygon making up the right triangle. The area of the red right triangle is

$$A = \left(\frac{1}{2}\right)(3)(8) \qquad\qquad A = 12$$

The area of the green right triangle is

$$A = \left(\frac{1}{2}\right)(2)(5) \qquad\qquad A = 5$$

The area of the blue irregular hexagon is

$$A = 8$$

The area of the yellow irregular hexagon is

$$A = 7$$

The area of the white square is

$$A = 1$$

So the right triangle's area is

$$A = 12 + 5 + 8 + 7 + 1 \qquad\qquad A = 33$$

What's going on?!? The area of the original right triangle was 32.5, then when we split it, it became 32, and now the area is 33, so there must be a reason!

Solving For the Change in Area

Let's look at the original right triangle drawn on a set of axes in Figure 34.

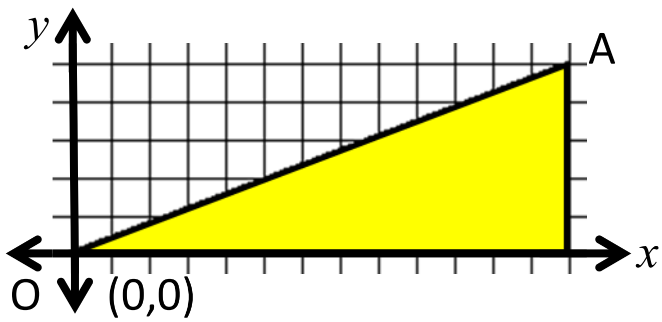



Figure 34. The main points of the figure are labeled O and A.

We will use the numbers 13 and 5 as the base and height. If we label the right triangle in Figure 34, it would be marked as shown in Figure 35.

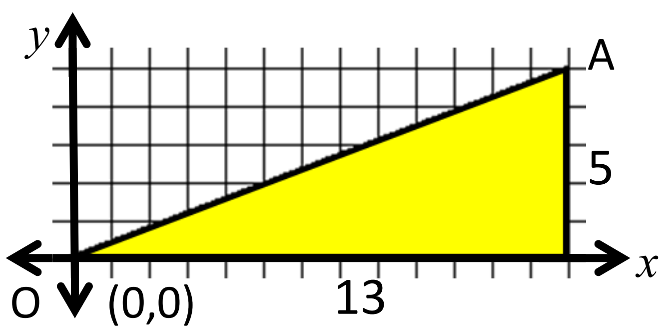



Figure 35. Notice that all the given side lengths are Fibonacci numbers.

Each of the points has its own coordinate on the axes. $\boldsymbol{O}$ is (0,0), and $\boldsymbol{A}$ is (13,5). Using these coordinates, let's find the slope of $\boldsymbol{OA}$.

$$\text{slope of } \boldsymbol{OA} = \frac{\mathbf{5}}{\mathbf{13}} \text{ or } \mathbf{0.\overline{384615}}$$

Let's look at the split right triangle drawn on a set of axes in Figure 36.

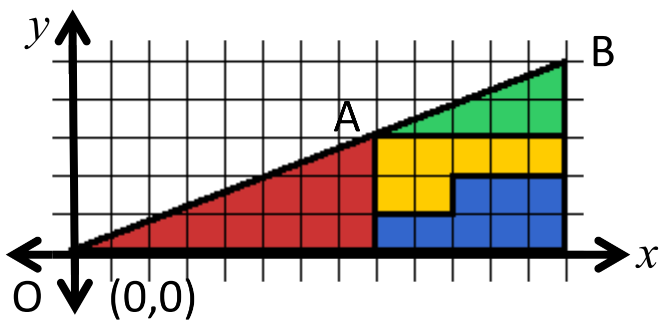



Figure 36. The main points of the figure are labeled O, A, and B.

We will use the numbers 13 and 5 as the base and height. If we label the triangle in Figure 36, it would be marked as shown in Figure 37.

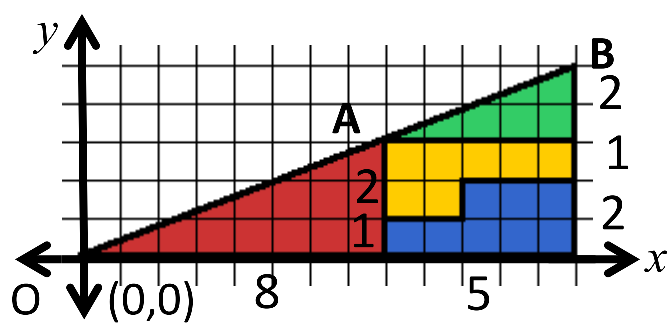



Figure 37. Notice that all the given side lengths are Fibonacci numbers.

Each of the points has its own coordinate on the axes. ***O*** is (0,0), ***A*** is (8,3), and ***B*** is (13,5). Using these coordinates, let's find the slope of ***OA*** and ***AB***.

$$\text{slope of } \boldsymbol{OA} = \frac{\mathbf{3}}{\mathbf{8}} \text{ or } \mathbf{0.375} \qquad \text{slope of } \boldsymbol{AB} = \frac{5-3}{13-8} = \frac{\mathbf{2}}{\mathbf{5}} \text{ or } \mathbf{0.4}$$

The slopes are not equal! That means that ***OAB*** is not a straight line! ***AB*** has a steeper slope than ***OA*** which creates a "kinked" line. In this case, the "kinked" line has a "dip" because the slope of ***AB*** is greater than the slope of ***OA***. This causes the difference in area that we discovered. Since it is a "dip" the area is decreasing from the original triangle by 0.5.

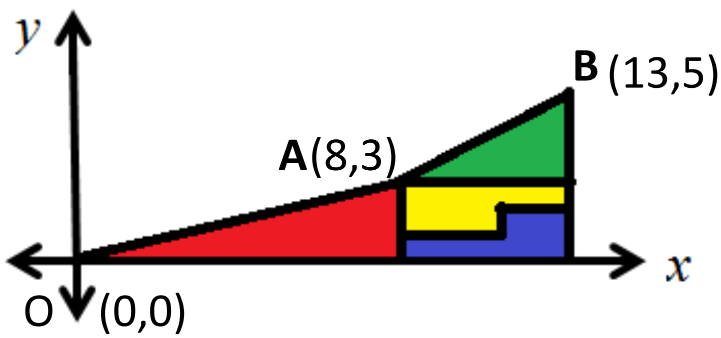



Figure 38. The slopes don't match which creates a "dip."

Let's look at the split right triangle drawn on a set of axes in Figure 39.

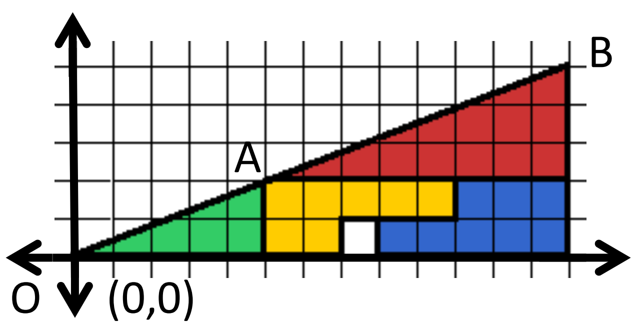



Figure 39. The main points of the figure are labeled O, A, and B.

We will use the numbers 13 and 5 as the base and height. If we label the triangle in Figure 39, it would be marked as shown in Figure 40.

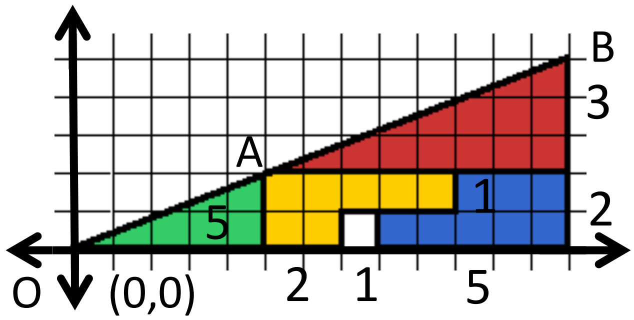



Figure 40. Notice that all the given side lengths are Fibonacci numbers.

Each of the points has its own coordinates on the axes. $\boldsymbol{O}$ is (0,0), $\boldsymbol{A}$ is (5,2), and $\boldsymbol{B}$ is (13,5). Using these coordinates, let's find the slope of $\boldsymbol{OA}$ and $\boldsymbol{AB}$.

$$\text{slope of } \boldsymbol{OA} = \frac{\mathbf{2}}{\mathbf{5}} \text{ or } \mathbf{0.4} \qquad \text{slope of } \boldsymbol{AB} = \frac{5-2}{13-5} = \frac{\mathbf{3}}{\mathbf{8}} \text{ or } \mathbf{0.375}$$

The slopes are not equal! That means that $\boldsymbol{OAB}$ is not a straight line! $\boldsymbol{OA}$ has a steeper slope than $\boldsymbol{AB}$ which creates a "kinked" line. In this case, the "kinked" line has a "hill" because the slope of $\boldsymbol{OA}$ is greater than the slope of $\boldsymbol{AB}$. This causes the difference in area that we discovered. Since it is a "hill," the area is increasing from the original triangle by 0.5.

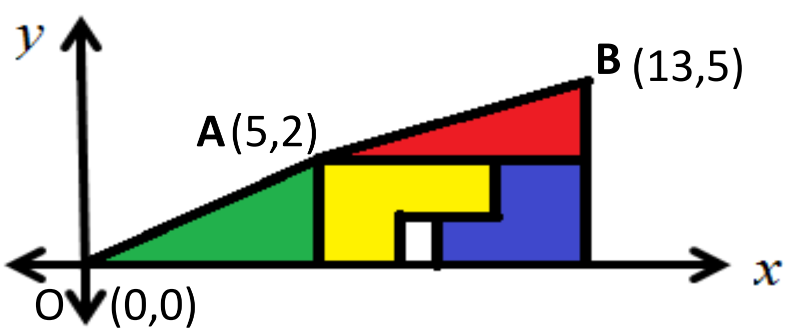



Figure 41. The slopes don't match which creates a "bump."

If ***OAB*** is not a straight line, then how come we did not notice it when we drew our graphs? Since the slopes are so close to one another, it is almost impossible to see the difference with a naked eye.

Relating Back to the Rectangle

If we look at the triangle closely, we can start to see a resemblance to the rectangle that we investigated. The triangle's arrangement can be simplified into two shapes: a right triangle and a trapezoid.

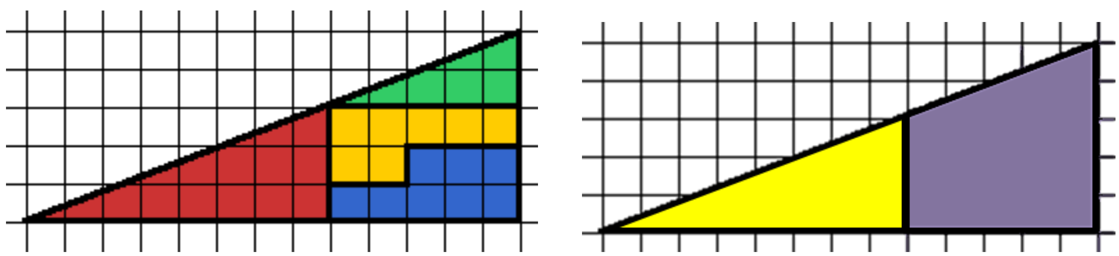


Figure 42. The triangle on the left is contains a right triangle and a trapezoid.

The triangle on the right in Figure 42 is the bottom half of the rectangle we were investigating that had a "gap."

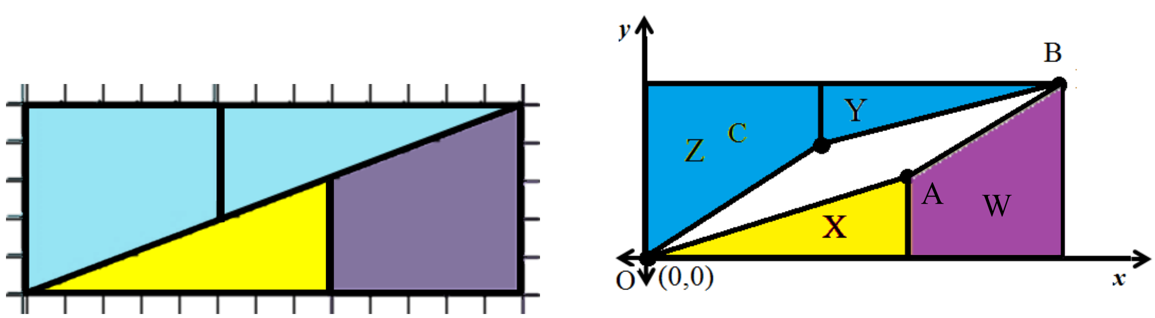



Figure 43. The triangle is part of the rectangle we investigated.

This further proves that the reason for the "dip" in the triangle in Figure 42 is the disagreement in slope.

RECOMMENDATIONS FOR FURTHER RESEARCH

We have just barely scratched the surface of the concepts in this paradox. In this paper, we discussed many things; one of them being an algebraic proof that ${f_{i+1}}^2 - f_i(f_{i+2}) = \pm 1$. While doing this proof, we just stated the formula $f_x = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^x - \left(\frac{1-\sqrt{5}}{2}\right)^x}{\sqrt{5}}$. We can extend our knowledge by creating a proof that this formula is true.

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